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NOTE ON IMPLICIT FUNCTIONS DEFINED BY
TWO EQUATIONS WHEN THE FUNCTIONAL
DETERMINANT VANISHES.

BY PROFESSOR W. R. LONGLEY.

(Read before the American Mathematical Society, December 30, 1908.)

1. *Introduction.* Consider the two equations

$$(1) \quad f(x_1, x_2, \dots, x_n; y, z) = 0, \quad g(x_1, x_2, \dots, x_n; y, z) = 0,$$

and suppose a single point solution

$$(2) \quad x_1 = a_1, x_2 = a_2, \dots, x_n = a_n; y = b, z = c,$$

is known. Under certain well-known conditions, of which one is the non-vanishing of the functional determinant $\partial(f, g)/\partial(y, z)$ at the point in question, we may affirm that equations (1) define uniquely the functions

$$(3) \quad y = \phi(x_1, x_2, \dots, x_n), \quad z = \psi(x_1, x_2, \dots, x_n)$$

in the neighborhood of the system of values (2). In general if the functional determinant vanishes, the functions (3) are multiple valued.* There are, however, certain exceptional cases in which the determination of y and z as functions $x_1, \dots, x_n$ is unique although the functional determinant vanishes. It is proposed in this paper to examine briefly some of these exceptional cases.

In a certain trivial way the corresponding exceptional cases exist also when we consider a single equation defining one dependent variable. Suppose an equation for the determination of y as a function of x has the form

$$f(x, y) \equiv x f_1(x, y) = 0.$$

*The case of analytic functions was treated by Professor G. A. Bliss in his Con Colloquium Lectures (1909). These lectures have not yet been ed.

Evidently $f = 0$ and $\partial f / \partial y = 0$ for $x = y = 0$, and the mental existence theorem is not applicable. But we may discard the factor x and if $f_1 = 0$, $\partial f_1 / \partial y \neq 0$ for $x = y = 0$ we may affirm the existence of a unique solution of the $y = \phi(x)$. Geometrically the locus defined by equation in the neighborhood of the origin consists of two branches; but of these is the curve $x = 0$, which is not expressible in the form $y = \phi(x)$. Again suppose the equation has the form

$$(5) \quad f(x, y) = f_1^2(x, y) = 0,$$

where $f_1(0, 0) = 0$. Here also $\partial f / \partial y = 0$ at the origin; we may consider the equation $f_1 = 0$ and, if $\partial f_1 / \partial y \neq 0$ for $x = y = 0$, we may affirm the existence of a unique solution of the form $y = \phi(x)$. Exceptional cases such as (4) and (5) are trivial because they are readily excluded by the usual assumption that f is irreducible in the argument y .

In the case of two equations we shall assume always that the first equation is irreducible; but this is not sufficient to exclude exceptional cases which are similar, geometrically, to those presented by equations (4) and (5). The point of view is shown by some simple examples involving only one independent variable. Consider the two equations

$$(6) \quad f \equiv y - x = 0, \quad g \equiv 2yz - x^2 = 0.$$

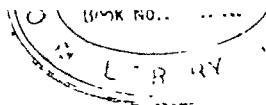
Both equations are satisfied for $x = y = z = 0$ and $\partial(f, g) / \partial(y, z) = 0$ at the origin. From the first equation $y = x$ and, when this value of y is substituted in the second, we get $x(2z - x) = 0$. This equation is of the form (4) and has a unique solution $z = \frac{1}{2}x$. Hence for values of x different from zero the system (6) defines uniquely the functions

$$(7) \quad y = \phi(x) \equiv x, \quad z = \psi(x) \equiv \frac{1}{2}x.$$

Geometrically the curve defined by (6) consists of two branches passing through the origin; but one branch is the z -axis, which is not expressible by equations of the form (7). Again consider the two equations

$$(8) \quad f \equiv z + 2y - 2x = 0, \quad g \equiv 2z + 4y - 4x + 2yz - x^2 = 0.$$

Both equations are satisfied for $x = y = z = 0$, and $\partial(f, g) / \partial(y, z) = 0$ at the origin. From the first equation



$z = 2x - 2y$ and, when this value is substituted in the second, we get $(2y - x)^2 = 0$. This equation is of the form (5) and has a unique solution $y = \frac{1}{2}x$. Hence for values of x different from zero the system (8) defines uniquely the functions

$$(9) \quad y = \phi(x) \equiv \frac{1}{2}x, \quad z = \psi(x) \equiv x.$$

Geometrically the curve defined by (8) consists of two coincident branches passing through the origin, that is, the two surfaces touch along the curve (9). Upon each surface the origin is an ordinary point.

The general method of investigating the solutions of equations (1), as illustrated by the preceding examples, is the following: Suppose one equation, say the first, can be solved by the fundamental theorem for one of the dependent variables, say y in terms of $z, x_1, \dots, x_n$. When this value of y is substituted in the second equation it may happen that the latter becomes reducible in $z, x_1, \dots, x_n$, and although the functional determinant vanishes the solution for z may be unique. For practical application it is convenient to formulate conditions upon f and g in order that this process shall lead to a unique result. Apparently no general formulation can be made, but it is possible to state theorems for special cases. Two of these theorems are given below.

2. In the theorem given in this section it is assumed that the functions f and g are real functions of real variables, and for convenience of reference the following statement of the fundamental theorem for two equations is given:

Fundamental Theorem. Consider the system of equations

$$(1) \quad f(x_1, x_2, \dots, x_n; y, z) = 0, \quad g(x_1, x_2, \dots, x_n; y, z) = 0,$$

and suppose a special solution

$$(2) \quad x_1 = a_1, x_2 = a_2, \dots, x_n = a_n; y = b, z = c,$$

is known. Suppose the functions f and g are continuous, and possess first partial derivatives which are continuous, in the neighborhood of the system of values (2). Suppose the functional determinant

$$\Delta = \begin{vmatrix} f_y & f_z \\ g_y & g_z \end{vmatrix}$$

does not vanish for the system of values (2).

Then there exists one and only one system of continuous functions of the form

$$A: \quad y = \phi(x_1, x_2, \dots, x_n), \quad z = \psi(x_1, x_2, \dots, x_n)$$

which satisfy equations (1) and the condition

$$b = \phi(a_1, a_2, \dots, a_n), \quad c = \psi(a_1, a_2, \dots, a_n).$$

Moreover the implicit functions A thus defined possess first partial derivatives which are continuous in the neighborhood of the system of values $x_1 = a_1, x_2 = a_2, \dots, x_n = a_n$.

The solution described in the conclusion of this theorem will be referred to as a solution of the type A . When the functional determinant vanishes the following theorem states what seem to be the simplest conditions of practical value in order that there shall exist a unique solution of the type A . It is assumed in this theorem that f and g possess partial derivatives of the first and second* orders which are continuous in the neighborhood of the system of values (2).

THEOREM I. Consider equations (1) and suppose

$$(a) \quad \begin{aligned} f(a_1, x_2, \dots, x_n; b, z) &\equiv 0, \\ g(a_1, x_2, \dots, x_n; b, z) &\equiv 0 \end{aligned} \quad (\text{in } x_2, \dots, x_n; z).$$

Suppose also

$$(b) \quad \Delta_1 = \begin{vmatrix} f_y & f_{x_1} \\ g_y & g_{x_1} \end{vmatrix} = 0, \quad (c) \quad \Delta_2 = \begin{vmatrix} 0 & f_y & f_{x_1} \\ f_y & f_{y^2} & f_{yx_1} \\ g_y & g_{y^2} & g_{yx_1} \end{vmatrix} \neq 0$$

for the system of values (2).

Then there exists one and only one solution of the type A .

To sketch the proof of this theorem we observe that condition (c) implies either $f_y \neq 0$ or $g_y \neq 0$. Assuming $f_y \neq 0$ we may apply the fundamental theorem for one equation to the first of equations (1) and obtain a solution for y . From the first of conditions (a) it follows that when $x_1 = a_1, y = b$ *identically* in $x_2, \dots, x_n, z$. Hence the solution for y has the form

$$(10) \quad y - b = (x_1 - a_1)h(x_1, x_2, \dots, x_n; z).$$

When this value of y is substituted in the second of equations (1) it follows from (a) that the equation assumes the form

*It is not necessary to assume the existence and continuity of all the partial derivatives of the second order. No attempt is being made here to reduce these conditions to a minimum.

$$(11) \quad (x_1 - a_1)g_1(x_1, x_2, \dots, x_n; z) = 0.$$

Now the appropriate computation shows that [for the system of values (2)] condition (b) is equivalent to the condition $g_1 = 0$, and (c) is equivalent to the condition $\partial g_1 / \partial z \neq 0$. Hence we may discard the factor $x_1 - a_1$ in equation (11) and apply the fundamental theorem for one equation to obtain a solution $z = \psi(x_1, x_2, \dots, x_n)$. When this value of z is substituted in (10) we have the final solution in the form

$$(12) \quad y = \phi(x_1, x_2, \dots, x_n), \quad z = \psi(x_1, x_2, \dots, x_n).$$

Each step in the process yields a unique result; hence the solution (12) is unique and of type A.

3. An example* of a system of equations of the type to which Theorem I is applicable may be found in a problem in the calculus of variations. The problem has been treated † in a paper by G. A. Bliss in the *BULLETIN*, volume 13 (1907), page 321, and the system referred to is (3), page 322. The equations to be solved for s and t are

$$(13) \quad f \equiv \phi(s, t) - r \cos \alpha = 0, \quad g \equiv \psi(s, t) - r \sin \alpha = 0,$$

where the functions ϕ and ψ satisfy the conditions

$$(14) \quad \begin{aligned} \phi(0, t) &\equiv 0, & \psi(0, t) &\equiv 0, \\ \phi_s(0, t) &\equiv \cos t, & \psi_s(0, t) &\equiv \sin t. \end{aligned}$$

Evidently a particular solution of equations (13) is

$$(15) \quad r = 0, \quad \alpha = k, \quad s = 0, \quad t = k,$$

where k is any constant.

For the system of values (15) the functional determinant

$$\Delta = \begin{vmatrix} \phi_s & \phi_t \\ \psi_s & \psi_t \end{vmatrix} = \begin{vmatrix} \cos k & 0 \\ \sin k & 0 \end{vmatrix} = 0,$$

and the fundamental theorem is not applicable. We apply then the hypothesis of Theorem I. The identities (14) show

* Other examples are to be found in papers by F. R. Moulton, "A class of periodic solutions, etc.," *Transactions*, vol. 7 (1906), p. 546, equations (16); and by W. R. Longley, "A class of periodic orbits, etc.," *Transactions*, vol. 8 (1907), p. 166, equations (15).

† See also Bolza, *Variationsrechnung*, p. 270.

that condition (a) is satisfied. Also

$$\Delta_1 = \begin{vmatrix} f_s & f_r \\ g_s & g_r \end{vmatrix} = \begin{vmatrix} \cos k & -\cos k \\ \sin k & -\sin k \end{vmatrix} = 0,$$

$$\Delta_2 = \begin{vmatrix} 0 & f_s & f_r \\ f_s & f_{ss} & f_{sr} \\ g_s & g_{ss} & g_{sr} \end{vmatrix} = \begin{vmatrix} 0 & \cos k & -\cos k \\ \cos k & -\sin k & 0 \\ \sin k & \cos k & 0 \end{vmatrix} = -\cos k.$$

Hence if $\cos k \neq 0$ the hypothesis of Theorem I is satisfied and we may affirm the existence of a unique solution of the type A. If $\cos k = 0$, then $\sin k \neq 0$ and the apparent difficulty is overcome by interchanging the notation f and g in equations (13).

4. If the functions f and g in Theorem I are analytic, the process of solving the first of equations (1) for y , substituting in the second, and solving for z shows that the solution for y and z is also analytic. The solution may be obtained, however, without going through this process. For simplicity this will be shown when there is only one independent variable x , and we will suppose that the equations are satisfied for $x = y = z = 0$. If f and g are regular at the origin, equations (1) may be written

$$(16) \quad f = \sum a_{ijk} x^i y^j z^k = 0, \quad g = \sum b_{ijk} x^i y^j z^k = 0.$$

For the system (16) the conditions (a), (b), and (c) are

$$(a) \quad a_{00k} = 0, \quad b_{00k} = 0 \quad (k = 1, 2, \dots),$$

$$(b) \quad \Delta_1 = \begin{vmatrix} a_{100} & a_{010} \\ b_{100} & b_{010} \end{vmatrix} = 0, \quad (c) \quad \Delta_2 = \begin{vmatrix} 0 & a_{010} & a_{100} \\ a_{010} & a_{011} & a_{101} \\ b_{010} & b_{011} & b_{101} \end{vmatrix} \neq 0.$$

The assumption (c) implies that either $a_{010} \neq 0$ or $b_{010} \neq 0$. Supposing a_{010} is not zero, we may without loss of generality take it equal to unity (by multiplying the series for f by $1/a_{010}$) and we may also take $\Delta_2 = 1$ (by multiplying the series for g by $1/\Delta_2$).

The solution to be determined has the form

$$(17) \quad y = \alpha_1 x + \alpha_2 x^2 + \dots, \quad z = \beta_1 x + \beta_2 x^2 + \dots.$$

When the values of y and z from (17) are substituted in (16) the coefficient of each power of x must vanish. Equating to

zero the coefficients of the first power of x we have

$$a_{100} + \alpha_1 = 0, \quad b_{100} + b_{010}\alpha_1 = 0.$$

From (b) it follows that these two equations are consistent and hence $\alpha_1 = -a_{100}$.

Equating to zero the coefficients of x^2 , we have

$$(18) \quad \begin{aligned} (a_{101} + a_{011}\alpha_1)\beta_1 + \alpha_2 + a_{200} + a_{020}\alpha_1^2 + a_{110}\alpha_1 &\equiv 0, \\ (b_{101} + b_{011}\alpha_1)\beta_1 + b_{010}\alpha_2 + b_{200} + b_{020}\alpha_1^2 + b_{110}\alpha_1 &= 0. \end{aligned}$$

When the value of α_1 is substituted in (18) it is found that the determinant of the coefficients of α_2 and β_1 is equal to $\Delta_2 (= 1)$. Hence equations (18) determine α_2 and β_1 uniquely as rational integral functions of the coefficients of (16) of order * less than 3.

Proceeding in this way, we may determine step by step the coefficients in the series (17). For suppose

$$\alpha_1, \dots, \alpha_{n-1}, \beta_1, \dots, \beta_{n-2}$$

have been determined as rational integral functions of the coefficients of (16) of order less than n . Equating to zero the coefficients of x^n we have

$$\begin{aligned} (a_{101} - a_{011}a_{100})\beta_{n-1} + \alpha_n + \phi_n &= 0, \\ (b_{101} - b_{011}a_{100})\beta_{n-1} + b_{010}\alpha_n + \psi_n &= 0, \end{aligned}$$

where ϕ_n, ψ_n denote rational integral functions of a_{ijk}, b_{ijk} of order less than $n+1$. The determinant of the coefficients of β_{n-1} and α_n is equal to unity and hence these quantities are determined uniquely as rational integral functions of the coefficients of (16) of order less than $n+1$.

5. A method of extending the conditions of Theorem I is suggested by considering the geometric interpretation. Suppose the equations involve three variables x, y, z , which will be interpreted as rectangular coordinates in space. The two surfaces $f=0, g=0$ then define a curve and we suppose the hypothesis of Theorem I is satisfied at a point which will be taken for the origin. Then from the conditions of the theorem it follows that the curve consists of two branches passing through the origin; but one of these branches is the z -axis and is not expressible by equations of the form $y = \phi(x), z = \psi(x)$.

*By the order of a coefficient is meant the number $i+j+k$.

It is apparent géométrically that if the curve consists of n branches passing through the origin, and if $n - 1$ of these branches are plane curves lying in the yz -plane, while one branch does not lie in this plane, then the equations $f=0, g=0$ will have one and only one solution of the form $y=\phi(x), z=\psi(x)$. When these geometric conditions are stated in analytic language we may formulate further theorems similar to Theorem I. Theorem II is given as an example.

THEOREM II. Consider equations (1) and suppose

$$\begin{aligned} & f(a_1, x_2, \dots, x_n; \lambda(z), z) \equiv 0, \\ (\alpha) \quad & \quad \quad \quad (\text{in } x_2, \dots, x_n; z) \\ & g(a_1, x_2, \dots, x_n; \lambda(z), z) \equiv 0, \end{aligned}$$

where the function $\lambda(z)$, which is substituted for y , is continuous and has a continuous first derivative in the neighborhood of $z=c$, and $b=\lambda(c)$. Suppose also

$$(\beta) \quad \Delta_1 = \begin{vmatrix} f_y & f_{x_1} \\ g_y & g_{x_1} \end{vmatrix} = 0,$$

$$(\gamma) \quad \Delta_2 = \begin{vmatrix} 0 & f_y & f_{x_1} \\ f_y & f_{yy} + \lambda' f_{yy} & f_{yx_1} + \lambda' f_{yx_1} \\ g_y & g_{yy} + \lambda' g_{yy} & g_{yx_1} + \lambda' g_{yx_1} \end{vmatrix} \neq 0,$$

for the system of values (2). (The derivative of λ is denoted by λ' .)

Then there exists one and only one solution of the type A.

In order to prove this theorem we set

$$(19) \quad y = \eta + \lambda(z)$$

in equations (1) and obtain the set

$$F(x_1, x_2, \dots, x_n; \eta, z) \equiv f(x_1, x_2, \dots, x_n; \eta + \lambda(z), z) = 0,$$

$$G(x_1, x_2, \dots, x_n; \eta, z) \equiv g(x_1, x_2, \dots, x_n; \eta + \lambda(z), z) = 0.$$

The condition (a) becomes

$$\begin{aligned} (a) \quad & F(a_1, x_2, \dots, x_n; 0, z) \equiv 0, \\ & G(a_1, x_2, \dots, x_n; 0, z) \equiv 0, \quad (\text{in } x_2, \dots, x_n; z) \end{aligned}$$

and the conditions (β) and (γ) become, respectively,

$$(b) \quad \begin{vmatrix} F_{\eta} & F_{x_1} \\ G_{\eta} & G_{x_1} \end{vmatrix} = 0, \quad (c) \quad \begin{vmatrix} 0 & F_{\eta} & F_{x_1} \\ F_{\eta} & F_{\eta z} & F_{zx_1} \\ G_{\eta} & G_{\eta z} & G_{zx_1} \end{vmatrix} \neq 0$$

for the system of values

$$x_1 = a_1, \quad x_2 = a_2, \quad \dots, \quad x_n = a_n; \quad \eta = 0, \quad z = c.$$

Hence the hypothesis of Theorem I is satisfied for the equations $F = 0$, $G = 0$, and we have a unique solution of the form

$$(20) \quad \eta = \eta(x_1, x_2, \dots, x_n), \quad z = \psi(x_1, x_2, \dots, x_n),$$

such that

$$0 = \eta(a_1, a_2, \dots, a_n), \quad c = \psi(a_1, a_2, \dots, a_n).$$

Substituting the values of η and z in (19), we have the required solution for y .

If there is only one independent variable x , the conditions of Theorem II, for the system of values $x = y = z = 0$, imply that the equations have the form

$$f(x, y, z) \equiv [y - \lambda(z)]f_1(x, y, z) + xf_2(x, y, z) = 0,$$

$$g(x, y, z) \equiv [y - \lambda(z)]g_1(x, y, z) + xg_2(x, y, z) = 0.$$

The curve defined by these equations consists of two branches passing through the origin. One branch is the curve $y = \lambda(z)$ in the yz -plane and is not expressible by equations of the form $y = \phi(x)$, $z = \psi(x)$.

SHEFFIELD SCIENTIFIC SCHOOL,
May, 1910.

STURM'S METHOD OF INTEGRATING

$$dx/\sqrt{X} + dy/\sqrt{Y} = 0.$$

BY PROFESSOR F. H. SAFFORD.

(Read before the American Mathematical Society, April 30, 1910.)

ONE of the simplest methods of obtaining the addition theorem for elliptic integrals of the first kind is based upon a method of integration which is usually referred to as Sturm's

method, and was thus communicated by Liouville in the *Comptes rendus*, 1856, No. 21, page 988. But on page 1087 of the same volume Liouville stated that the method was due to Despeyroux, and in the *Cours d'analyse* of Ch. Sturm, Paris, 1877, volume 2, page 340, the method is given in a note "par M. Despeyroux." The object of the present paper is to discuss certain details of Sturm's method and the extent to which it may be employed, without considering the addition theorems. The method is applicable to special types of the equation

$$(1) \quad dx/\sqrt{f(x)} + dy/\sqrt{f(y)} = 0,$$

and since the variables in (1) are separated it can be integrated first term by term. By Sturm's method a second integral of (1) is sometimes obtainable by clearing of fractions and in most cases using also an integrating factor. Integration by parts is performed on each term of the new form of (1) and the two indicated integrations at this stage are to be combined under one integral sign; then if possible by the use of (1) this integrand is to be made to vanish. When this method is successful a comparison of the resulting integral with that first obtained leads to an addition theorem. Illustrations may be found in Byerly, *Integral Calculus*, page 235, and Dürège, *Elliptische Funktionen*, 4th edition, page 101.

So far as the writer has discovered, in all cases where this method has been used the form of $f(x)$ has been considered known, and from the types x^2 , $1-x^2$, $(1+x^2)^2$, $(1-x^2)(1-k^2x^2)$ follow addition theorems for $\log x$, $\sin^{-1}x$, $\tan^{-1}x$, and $\operatorname{sn}^{-1}x$ respectively. The last two types require an additional integrating factor but the first two do not.

The following is from Schlömilch, *Compendium der höheren Analysis*, volume 2, page 326: "Es sei * * *

$$(2) \quad dx/\sqrt{1-x^2} + dy/\sqrt{1-y^2} = 0$$

oder

$$(3) \quad \sqrt{1-y^2}dx + \sqrt{1-x^2}dy = 0,$$

mithin

$$(4) \quad \int \sqrt{1-y^2}dx + \int \sqrt{1-x^2}dy = \text{const.}$$

Bei theilweiser Integration ist ferner

$$(5) \quad \int \sqrt{1-y^2} dx = \sqrt{1-y^2} x + \int \frac{xy dy}{\sqrt{1-y^2}},$$

$$\int \sqrt{1-x^2} dy = \sqrt{1-x^2} y + \int \frac{yx dx}{\sqrt{1-x^2}}$$

mithin durch Addition

$$(6) \quad x\sqrt{1-y^2} + y\sqrt{1-x^2} + \int xy \left[\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} \right] = \text{const.}$$

Wegen Nro. (2) verschwindet das Integral und es bleibt einfacher

$$(7) \quad x\sqrt{1-y^2} + y\sqrt{1-x^2} = c, \quad * * *.$$

Though (7) is an integral of (2) it should be noticed that (3) is not an exact differential, and the integral in (6) for the analogous case $f(x) = x^2$ is a constant and does not vanish.

Schröter in the *Zeitschrift für Mathematik und Physik*, volume 17 (1872), page 508, avoids the difficulty by an inverse method which, applied to the example above, gives

$$(8) \quad d[x\sqrt{1-y^2} + y\sqrt{1-x^2}]$$

$$= [\sqrt{1-y^2} dx + \sqrt{1-x^2} dy] - xy \left[\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} \right]$$

$$= \left[\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} \right] [\sqrt{1-x^2}\sqrt{1-y^2} - xy].$$

Thus from (2) equation (8) becomes

$$(9) \quad d[x\sqrt{1-y^2} + y\sqrt{1-x^2}] = 0,$$

whence the integral (7) follows. In this case no additional integrating factor is necessary.

The use of Sturm's method in integrating

$$(10) \quad \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} + \frac{dy}{\sqrt{(1-y^2)(1-k^2y^2)}} = 0$$

has been called "elegant" by several writers who, however, have felt that the form of the integrating factor, after clearing of fractions, i. e., $1/(1-k^2x^2y^2)$, is not obvious from the form of equation (10).

Schröter makes changes of variables in (10) and then shows how to determine the factor. But in the present paper the factor will be expressed, under certain sufficient conditions, as a

function of $f(x)$ and $f(y)$ in the general type, equation (1), and then the corresponding form of $f(x)$ will be determined.

From (1), after supplying the unknown factor $F(x, y)$,

$$(11) \quad \frac{\sqrt{f(y)}dx}{F} + \frac{\sqrt{f(x)}dy}{F} = 0.$$

The use of Sturm's method in its modified form gives

$$\begin{aligned} & d \left[\frac{x\sqrt{f(y)}}{F} + \frac{y\sqrt{f(x)}}{F} \right] \\ &= [x\sqrt{f(y)} + y\sqrt{f(x)}] \left[\frac{\frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy}{-F^2} \right] \\ &+ dx \left[\frac{\sqrt{f(y)} + \frac{yf'(x)}{2\sqrt{f(x)}}}{F} \right] + dy \left[\frac{\sqrt{f(x)} + \frac{xf'(y)}{2\sqrt{f(y)}}}{F} \right] \\ (12) \quad &= \frac{dx}{\sqrt{f(x)}} \left[-\frac{x\sqrt{f(x)}\sqrt{f(y)}}{F^2} \frac{\partial F}{\partial x} - \frac{yf'(x)}{F^2} \frac{\partial F}{\partial x} \right. \\ &\quad \left. + \frac{\sqrt{f(x)}\sqrt{f(y)}}{F} + \frac{yf'(x)}{2F} \right] \\ &+ \frac{dy}{\sqrt{f(y)}} \left[-\frac{y\sqrt{f(x)}\sqrt{f(y)}}{F^2} \frac{\partial F}{\partial y} - \frac{xf'(y)}{F^2} \frac{\partial F}{\partial y} \right. \\ &\quad \left. + \frac{\sqrt{f(x)}\sqrt{f(y)}}{F} + \frac{xf'(y)}{2F} \right]. \end{aligned}$$

The last member of (12) will vanish in virtue of (1), provided the coefficients of $dx/\sqrt{f(x)}$ and $dy/\sqrt{f(y)}$ are equal, and the sufficient conditions are these :

$$(13) \quad x \frac{\partial F}{\partial x} = y \frac{\partial F}{\partial y}.$$

$$(14) \quad yf(x) \frac{\partial F}{\partial x} - \frac{yFf'(x)}{2} = xf(y) \frac{\partial F}{\partial y} - \frac{xFf'(y)}{2}.$$

From (13) it follows that F is a function of the product xy , which is an important result and permits the reduction of (14) to

$$(15) \quad 2x^2 F'f(y) - xFf'(y) = 2y^2 F'f(x) - yFf'(x),$$

whence

$$(16) \quad \frac{2F'}{F} = \frac{xf''(y) - yf''(x)}{x^2f(y) - y^2f(x)}.$$

Since F is a function of xy , the second member of (16) must also be a function of xy , and for proper forms of $f(x)$, F may be found from (16) by integration. An illustration is $f(x) = (1 - x^2)(1 - k^2x^2)$, then F is $c(1 - k^2x^2y^2)$.

The determination of the form of $f(x)$ is the next important step. Let the second member of (16) be called V ; then employing the condition that V shall be a function of xy , i. e.,

$$(13a) \quad x \frac{\partial V}{\partial x} = y \frac{\partial V}{\partial y},$$

the result after some reduction is

$$(17) \quad \begin{aligned} & \frac{f(x)f''(x)}{x^3} - \frac{f(y)f''(y)}{y^3} + \frac{f(x)f''(x)}{x^3} - \frac{f(y)f''(y)}{y^3} \\ & + \frac{f''(y)}{y^3} - \frac{f''(x)}{x^3} + \frac{3f'(x)f(y)}{xy^2} - \frac{3f'(y)f(x)}{yx^2} \\ & + \frac{f'(x)f''(y)}{x^2} - \frac{f'(y)f''(x)}{y^2} = 0. \end{aligned}$$

When (17) is differentiated partially with respect to x and y in succession, the first six terms vanish and the remaining four give after reduction

$$(18) \quad \begin{aligned} & \left[\frac{f''(x)}{x^2} - \frac{2f'(x)}{x^3} \right] \left[f'''(y) - \frac{3f''(y)}{y} + \frac{3f'(y)}{y^2} \right] \\ & = \left[\frac{f''(y)}{y^2} - \frac{2f'(y)}{y^3} \right] \left[f'''(x) - \frac{3f''(x)}{x} + \frac{3f'(x)}{x^2} \right]. \end{aligned}$$

The variables in (18) may now be separated by division, and since the result is that a function of x equals a function of y , each function is a constant, i. e.,

$$(19) \quad \frac{f'''(x) - \frac{3f''(x)}{x} + \frac{3f'(x)}{x^2}}{\frac{f''(x)}{x^2} - \frac{2f'(x)}{x^3}} = 4(a^2 - 1).$$

The form of the constant is chosen so that later results will be more simple, and the general solution of (19) is

$$(20) \quad f(x) = \alpha x^{2(1-a)} + \beta x^2 + \gamma x^{2(1+a)}.$$

From (16) and (20),

$$(21) \quad F = C_1(\alpha x^{1-a} y^{1-a} - \gamma x^{1+a} y^{1+a}).$$

Having satisfied the conditions expressed by (13) and (14), the last member of (12) vanishes and the first member becomes an exact differential, whence integration and substitution of f and F from (20) and (21) give

$$(22) \quad \frac{x\sqrt{(\alpha y^{2(1-a)} + \beta y^2 + \gamma y^{2(1+a)})} + y\sqrt{(\alpha x^{2(1-a)} + \beta x^2 + \gamma x^{2(1+a)})}}{\alpha x^{1-a} y^{1-a} - \gamma x^{1+a} y^{1+a}} = C,$$

which is an integral of

$$(23) \quad \frac{dx}{\sqrt{(\alpha x^{2(1-a)} + \beta x^2 + \gamma x^{2(1+a)})}} + \frac{dy}{\sqrt{(\alpha y^{2(1-a)} + \beta y^2 + \gamma y^{2(1+a)})}} = 0.$$

Making $a = 1$, or replacing x^a by x , and y^a by y , which may be shown to have the same effect, gives what may be considered the standard forms

$$(24) \quad \frac{x\sqrt{(\alpha + \beta y^2 + \gamma y^4)} + y\sqrt{(\alpha + \beta x^2 + \gamma x^4)}}{\alpha - \gamma x^2 y^2} = C,$$

$$(25) \quad \frac{dx}{\sqrt{(\alpha + \beta x^2 + \gamma x^4)}} + \frac{dy}{\sqrt{(\alpha + \beta y^2 + \gamma y^4)}} = 0.$$

The denominator in (24) is the integrating factor sought, and is a simple function of α and γ in (25), being independent of β . By giving proper values to α , β , and γ in (24) and (25) addition formulas may be obtained for the logarithm, inverse sine and tangent (both circular and hyperbolic), and elliptic integrals of the first kind. An illustration of the use of the apparently more general forms (22) and (23) is obtained by taking $a = \frac{1}{2}$, then $f(x)$ is $\alpha x + \beta x^2 + \gamma x^3$, which includes $x(1-x)$ ($1-\kappa x$), favored by Klein as a canonical form in elliptic integrals.

In Schröter's article (loc. cit.) the starting point is

$$(26) \quad d\phi/\Delta\phi + d\psi/\Delta\psi = 0 \quad [\Delta\phi = \sqrt{1 - k^2 \sin^2 \phi}].$$

Thus the form of $f(\phi)$ is assigned in advance, and (26) is inte-

grated by Sturm's method as three distinct problems, each being obtained by a change of variables. Although the Legendrian notation is practically adhered to, the respective substitutions are equivalent to

$$(27) \quad 1 - k^2 \sin^2 \phi = x^2, \quad \cos^2 \phi = x^2, \quad \sin^2 \phi = x^2.$$

The respective values of $f(x)$ are equivalent to

$$(28) \quad (1 - x^2)(x^2 - k^2), (1 - x^2)(k^2 + k^2 x^2), (1 - x^2)(1 - k^2 x^2).$$

Schröter's values of F' are deduced for each of the three cases independently, by expressing the sufficient conditions for the vanishing of coefficients in his equation corresponding to (12) above. His results are obtainable at once by using his respective values of $f(x)$ from (28) in (25) and (24), and this affords a concrete illustration of the theorem which these two equations express.

UNIVERSITY OF PENNSYLVANIA,
April, 1910.

A PROPERTY OF A SPECIAL LINEAR SUBSTITUTION.

BY PROFESSOR F. R. SHARPE.

LET ξ_i denote homogeneous line coordinates * which satisfy the quadratic identity

$$(1) \quad \Sigma \xi_i^2 = 0.$$

The condition that two lines ξ, ξ' intersect is

$$(2) \quad \Sigma \xi_i \xi'_i = 0.$$

The equation of a linear complex is of the form

$$(3) \quad \Sigma a_i \xi_i = 0.$$

It is clear from the above equations that, when

$$(4) \quad \Sigma a_i^2 = 0,$$

the lines of (3) all intersect the line a ; (3) is then called a special complex.

* Jessop: Treatise on the line complex, Arts 9, 15-20.

If

$$\lambda = -\frac{2\Sigma a_i x_i}{\Sigma a_i^2},$$

then

$$\Sigma(x_i + \lambda a_i)^2 = \Sigma x_i^2.$$

Hence

$$(5) \quad \Sigma x'_i \xi_i = \Sigma(2\Sigma a_i x_i \cdot a_i - \Sigma a_i^2 \cdot x_i) \xi_i = 0$$

is a special complex if

$$(6) \quad \Sigma x_i \xi_i = 0$$

is a special complex.

The linear complex (5) may be termed the transform of (6) through (3).

Two complexes x and y are said to be in involution if

$$(7) \quad \Sigma x_i y_i = 0.$$

When the two complexes are transformed through a

$$(8) \quad \Sigma x' y' = (\Sigma a_i^2)^2 \cdot \Sigma x_i y_i = 0.$$

Hence the transforms are also in involution. Consider the transformations

$$(A) \quad x'_i = 2\Sigma a_i x_i \cdot a_i - \Sigma a_i^2 \cdot x_i,$$

$$(B) \quad x''_i = 2\Sigma b_i x'_i \cdot b_i - \Sigma b_i^2 \cdot x'_i,$$

which transform x through the complexes a and b respectively, and

$$(A') \quad x'''_i = 2\Sigma(sa_i + tb_i)x_i \cdot (sa_i + tb_i) - \Sigma(sa_i + tb_i)^2 \cdot x_i,$$

$$(B') \quad x''''_i = 2\Sigma(s'a_i + t'b_i)x'_i \cdot (s'a_i + t'b_i) - \Sigma(s'a_i + t'b_i)^2 \cdot x'_i,$$

which transform x through two complexes of the pencil defined by a and b . These transformations have the property that if A , B , and A' are given, then B' can be uniquely determined so that either

$$(9) \quad AB = A'B'$$

or

$$(9)^* \quad AB = B'A'.$$

This can be proved very simply as follows. The coefficient of x_j in AB is

$$4a_j b_i \Sigma a_i b_i - 2a_j a_i \Sigma b_i^2 - 2b_j b_i \Sigma a_i^2.$$

If (9) is satisfied, the corresponding coefficients must have a common ratio ρ , so that

$$(10) \quad x_i'' \equiv \rho x_i^{IV}.$$

Hence

$$\rho = \frac{\text{coefficient of } x_i \text{ in } x_i''}{\text{coefficient of } x_i \text{ in } x_i^{IV}} = \frac{\text{coefficient of } x_i \text{ in } x_j''}{\text{coefficient of } x_i \text{ in } x_j^{IV}}.$$

Subtracting the numerators and denominators, we find after a little reduction

$$(11) \quad \rho = \frac{\Sigma a_i b_i}{(st' - s't) \Sigma (sa_i + tb_i, s'a_i + t'b_i)}.$$

Similarly by addition we find

$$(12) \quad \rho = \frac{1}{(st' - s't)^2}.$$

From (11) and (12) it follows that

$$(13) \quad \Sigma (sa_i + tb_i, s'a_i + t'b_i) = (st' - s't) \Sigma a_i b_i,$$

that is

$$(13)^* \quad ss' \Sigma a_i^2 + 2s't \Sigma a_i b_i + tt' \Sigma b_i^2 = 0.$$

Solving, we have

$$(14) \quad \frac{s'}{t'} = \frac{-t \Sigma b_i^2}{s \Sigma a_i^2 + 2t \Sigma a_i b_i}.$$

If we take (9)* instead of (9), then (13) becomes

$$(15) \quad \Sigma (sa_i + tb_i, s'a' + t'b_i) = (s't - st') \Sigma a_i b_i,$$

and (14) becomes

$$(16) \quad \frac{s'}{t'} = \frac{t \Sigma b_i^2 + 2s \Sigma a_i b_i}{-s \Sigma a_i^2}.$$

The equations (13) and (15) are both included in

$$(17) \quad \{\Sigma (sa_i + tb_i, s'a_i + t'b_i)\}^2 = (st' - s't)^2 \{\Sigma^2 a_i b_i\}^2,$$

but

$$(18) \quad \begin{aligned} & \{\Sigma (sa_i + tb_i, s'a_i + t'b_i)\}^2 - \Sigma (sa_i + tb_i)^2 \cdot \{\Sigma (s'a_i + t'b_i)^2 \\ & \quad = (st' - s't)^2 \{(\Sigma a_i b_i)^2 - \Sigma a_i^2 \cdot \Sigma b_i^2\}. \end{aligned}$$

Hence (17) may be written in the form

$$(19) \quad \frac{\{\Sigma(sa_i + tb_i, s'a_i + t'b_i)\}^2}{\Sigma(sa_i + tb_i)^2 \cdot \Sigma(s'a_i + t'b_i)^2} = \frac{\{\Sigma a_i b_i\}^2}{\Sigma a_i^2 \cdot \Sigma b_i^2}.$$

It is important to notice that the roots of the quadratic (19) in s'/t' are rationally separable and are given by (14) and (16), which are the solutions of (9) and (9)* respectively.

It remains to show that

$$(20) \quad \rho = \frac{\text{coefficient of } x_i \text{ in } x_i'''}{\text{coefficient of } x_i \text{ in } x_i^{IV}}.$$

The coefficients are similar to those which were used in (11) and (12), but have an extra term

$$(21) \quad \Sigma a_i^2 \cdot \Sigma b_i^2 \quad \text{and} \quad \Sigma(sa_i + tb_i)^2 \cdot \Sigma(s'a_i + t'b_i)^2$$

respectively. From (17) and (18) it follows that the terms (21) are also in the ratio 1 to $(st' - s't)^2$. This completes the proof of the property. The property may be expressed in terms of the geometry of the sphere.*

If we regard the x_i as point coordinates in hyperspace, then (5) shows that the quadratic variety

$$\Sigma x_i^2 = 0$$

is invariant under the transformation and (8) shows that polarity is preserved. If in (13)* we put

$$s = 0, \quad t' = 0,$$

we find that the condition that A and B are commutative is

$$\Sigma a_i b_i = 0.$$

Hence it follows that A and B are commutative when a and b are in involution.

CORNELL UNIVERSITY,
March, 1910.

* P. F. Smith, *Transactions*, vol. 1 (1900), p. 378, footnote. The statement in the footnote overlooks the distinction between our equations (14), (16).

ON THE FACTORIZATION OF INTEGRAL FUNCTIONS WITH p -ADIC COEFFICIENTS.

BY PROFESSOR L. E. DICKSON.

(Read before the American Mathematical Society, September 6, 1910.)

1. If $F(x)$ is an integral function of degree n with integral p -adic coefficients, then for any integer k we have a congruence

$$(1) \quad F(x) \equiv F^{(k)}(x) = F_0(x) + pF_1(x) + p^2F_2(x) \\ + \cdots + p^kF_k(x) \pmod{p^{k+1}},$$

in which each $F_i(x)$ is an integral function of degree $\leq n$ with coefficients belonging to the set $0, 1, \dots, p-1$. The function $F^{(k)}(x)$ is called the convergent of rank k of $F(x)$. If

$$(2) \quad F(x) = f(x) \cdot g(x) \pmod{p},$$

in which the factors are integral functions with integral p -adic coefficients, then for any integer k we obviously have

$$(3) \quad F^{(k)}(x) \equiv f^{(k)}(x) \cdot g^{(k)}(x) \pmod{p^{k+1}}.$$

The following converse theorem plays a fundamental rôle in Hensel's new theory of algebraic numbers: * If

$$(4) \quad F(x) \equiv f_0(x) \cdot g_0(x) \pmod{p^{s+1}}$$

for $\dagger s+1 > 2\rho$, where ρ is the order of the resultant R of $f_0(x)$ and $g_0(x)$, then $F(x)$ is the product (2) of two integral functions with integral p -adic coefficients whose convergents of rank $s-\rho$ are $f_0(x)$ and $g_0(x)$.

Hensel's proof is in effect a process to construct the successive convergents of $f(x)$ and $g(x)$. Each step of the process requires the solution of a linear equation in two unknowns with p -adic coefficients. The object of this note is to furnish a decidedly simpler process, which dispenses with these linear

* Hensel, *Theorie der algebraischen Zahlen*, Leipzig, Teubner, 1908, p. 71.

† This condition is satisfied if $s = \delta$, where δ is the order of the discriminant of $F(x)$. Hence we obtain as a corollary the theorem of Hensel, page 68.

equations, and requires only the solution of a single linear congruence.

By the remark of Hensel, bottom of page 64, we may assume that the leading coefficients in $F(x)$, $f_0(x)$, and $g_0(x)$ are powers of p , so that $F - f_0 g_0$ is of degree less than n .

2. For the sake of simplicity, we first establish the theorem for the important case $s = \rho = 0$: If $F(x) = F_0(x) + pF_1(x) + \dots$, in which the coefficient of x^n in F_0 is unity and $F_i (i > 0)$ is of degree less than n , and if

$$(5) \quad F_0(x) \equiv f_0(x) \cdot g_0(x) \pmod{p},$$

in which f_0 and g_0 are integral functions of degrees μ and ν respectively, with integral coefficients, while f_0 and g_0 have no common factor modulo p , then integral functions $f_i(x)$ of degrees $< \mu$ and $g_i(x)$ of degrees $< \nu$ with integral coefficients can be so determined that

$$f = f_0 + pf_1 + p^2f_2 + \dots, \quad g = g_0 + pg_1 + p^2g_2 + \dots$$

have as their product $F(x)$. It is meant by the last statement that congruence (3) holds for every integer k .

Since f_0 and g_0 have no common factor, integral functions $a(x)$ and $b(x)$ with integral coefficients can be determined (for example, by Euclid's process) such that

$$(6) \quad af_0 + bg_0 \equiv 1 \pmod{p}.$$

By (5), $F_0 - f_0 g_0$ is the product of p by an integral function with integral coefficients which may be designated $L_1(x) - F_1(x)$,

$$(7) \quad F_0 - f_0 g_0 = p(L_1 - F_1).$$

By the remark at the end of § 1, L_1 like F_1 is of degree less than n . The congruence

$$F_0 + pF_1 \equiv (f_0 + pf_1)(g_0 + pg_1) \pmod{p^2}$$

is equivalent, in view of (7), to

$$(8) \quad L_1 \equiv f_0 g_1 + g_0 f_1 \pmod{p}.$$

In view of (6), every set of solutions is given by

$$(9) \quad f_1 \equiv bL_1 - \rho_1 f_0, \quad g_1 \equiv aL_1 + \rho_1 g_0.$$

We choose $\rho_1(x)$ so that the degree of f_1 shall be less than the degree μ of f_0 . Then the final term of (8) is of degree $< \mu + \nu$,

so that the degree of g_1 is $< \nu$. Hence the required functions f_1 and g_1 are given by (9).

To make the general step by induction, suppose that, for $i=1, \dots, k$, functions f_i of degrees $< \mu$ and g_i of degrees $< \nu$ have been determined so that (3) holds. Hence we may set

$$(10) \quad F^{(k)} = f^{(k)}g^{(k)} + p^{k+1}(L_{k+1} - F'_{k+1}),$$

where L_{k+1} is of degree less than n . Since

$$(11) \quad \begin{aligned} F^{(k+1)} &= F^{(k)} + p^{k+1}F'_{k+1}, & f^{(k+1)} &= f^{(k)} + p^{k+1}f_{k+1}, \\ g^{(k+1)} &= g^{(k)} + p^{k+1}g_{k+1}, \end{aligned}$$

the condition for the congruence

$$F^{(k+1)} \equiv f^{(k+1)}g^{(k+1)} \pmod{p^{k+2}}$$

becomes, upon applying (10),

$$L_{k+1} \equiv f^{(k)}g_{k+1} + g^{(k)}f_{k+1} \equiv f_0g_{k+1} + g_0f_{k+1} \pmod{p}.$$

The general set of solutions is

$$f_{k+1} \equiv bL_{k+1} - \rho_{k+1}f_0, \quad g_{k+1} \equiv aL_{k+1} + \rho_{k+1}g_0.$$

We determine ρ_{k+1} so that the degree of f_{k+1} shall be less than the degree μ of f_0 ; then the degree of g_{k+1} will be less than the degree ν of g_0 . Since the induction is complete, our theorem is proved.

3. We readily deduce recursion formulas for the functions L_i . We have proved that functions L_i and ρ_i can be determined so that

$$(12) \quad f_i = bL_i - \rho_i f_0, \quad g_i = aL_i + \rho_i g_0 \quad (i \geq 1)$$

give functions f_i of degrees $< \mu$ and g_i of degrees $< \nu$ for which congruence (3) holds for every k . From (6),

$$(13) \quad af_0 + bg_0 = 1 + p\lambda(x).$$

In (10) we replace $F^{(k)}, f^{(k)}, g^{(k)}$ by the values obtained by replacing k by $k-1$ in (11) and then replace $F^{(k-1)}$ by the value obtained by replacing k by $k-1$ in (10). After deleting the factor p^k , we get

$$L_k = f_k g^{(k-1)} + g_k f^{(k-1)} + p^k f_k g_k + pL_{k+1} - pF'_{k+1}.$$

In the terms $f_k g_0 + g_k f_0$ we replace f_k and g_k by their values

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(12) and apply (13). After deleting the factor p we get

$$(14) \quad L_{k+1} = F'_{k+1} - \lambda L_k - f_k[g]_{k-1} - g_k[f]_{k-1} - p^{k-1}f_k g_k$$

in which we have employed the abbreviation

$$(15) \quad [z]_i = z_i + pz_2 + p^2z_3 + \dots + p^{i-1}z_n \quad [z]_0 = 0.$$

If in (10) we set

$$F^{(k)} = F'_0 + p[F]_k, \quad f^{(k)} = f_0 + p[f]_k, \quad g^{(k)} = g_0 + p[g]_k \\ \gamma = (F'_0 - f_0 g_0)/p,$$

and then delete the factor p , we get

$$f_0[g]_k + g_0[f]_k + p^k L_{k+1} = \gamma + [F]_k + p^k F'_{k+1} - p[f]_k[g]_k.$$

By (12), (15), the sum of the first two terms equals

$$\sum_{i=1}^k p^{i-1}(af_0 + bg_0)L_i = (1 + p\lambda) \sum_{i=1}^k p^{i-1}L_i = (1 + p\lambda)[L]_k.$$

Hence we obtain the formula

$$(16) \quad [L]_{k+1} = \gamma + [F]_{k+1} - p\lambda[L]_k - p[f]_k[g]_k.$$

It is also easy to establish this formula by induction.

4. To prove the more general theorem of § 1, we apply the method of § 2 with congruence (6) replaced by

$$(17) \quad af_0 + bg_0 \equiv p^p \pmod{p^{p+1}}.$$

Since the resultant of $f_0(x)$ and $g_0(x)$ is divisible by p^p , but by no higher power of p , solutions $a(x)$ and $b(x)$ of (17) can be determined by the method given by Hensel on pages 62, 63. In view of (4),

$$F^{(s+1)} - f_0 g_0 = p^{s+1} L_1,$$

where L_1 is of degree less than n . Then the congruence

$$(18) \quad F^{(s+1)} \equiv (f_0 + pf_1)(g_0 + pg_1) \pmod{p^{s+2}}$$

is satisfied if we take

$$f_1 \equiv p^{s-p}(bL_1 - \rho_1 f_0), \quad g_1 \equiv p^{s-p}(aL_1 + \rho_1 g_0) \pmod{p^{s+1}},$$

since by (17)

$$(20) \quad f_0 g_1 + f_1 g_0 \equiv p^s L_1 \pmod{p^{s+1}},$$

and $p^2 f_1 g_1$ is divisible by p^t , where $t = 2 + 2s - 2\rho > s + 1$. We choose $\rho_1(x)$ so that the degree of $f_1(x)$ shall be less than the degree of $f_0(x)$; then by (20) the degree of $g_1(x)$ will be less than the degree of $g_0(x)$.

Similarly, if in accord with (18) we set

$$F^{(s+1)} - (f_0 + pf_1)(g_0 + pg_1) = p^{s+2}(L_2 - F_{s+2}),$$

the congruence

$$F^{(s+2)} \equiv (f_0 + pf_1 + p^2 f_2)(g_0 + pg_1 + p^2 g_2) \pmod{p^{s+3}}$$

is satisfied if we take

$$f_2 \equiv p^{s-\rho}(bL_2 - \rho_2 f_0), \quad g_2 \equiv p^{s-\rho}(aL_2 + \rho_2 g_0) \pmod{p^{s+1}}.$$

The general step in the proof may now be made as in § 2.

HENSEL'S THEORY OF ALGEBRAIC NUMBERS.

Theorie der Algebraischen Zahlen. Von KURT HENSEL.
Erster Band. Leipzig and Berlin, Teubner, 1908. xi + 346 pp.

IN the theory of functions one may investigate an analytic function in the neighborhood of a point $z = a$ by means of a power series in $z - a$. In arithmetic one usually employs only developments according to the fixed base 10. The author undertakes in the present work to introduce a corresponding mobility into arithmetic and algebra by employing expansions of numbers into power series in an arbitrary prime number p .

A positive integer D can be expressed in one and but one way in the form

$$D = d_0 + d_1 p + \dots + d_k p^k,$$

in which each d_i is one of the integers $0, 1, \dots, p - 1$. This equation will be said to define the representation of D as a p -adic number, for which the following symbol will be employed:

$$D = d_0, d_1 d_2 \dots d_k (p).$$

For example,

$$14 = 2 + 3 + 3^2 = 2, 11 \quad (3), \quad 216 = 2 \cdot 3^3 + 2 \cdot 3^4 = 0, 0022 \quad (3).$$

The sum of two such p -adic numbers is readily expressed as a p -adic number. For example,

$$0, 0022 + 0, 1021 = 0, 10111 \quad (3).$$

When $A \geq B$, the difference $A - B$ is expressible as a p -adic number. Thus, if $A = 216$, $B = 138$, $p = 3$, we have

$$0,0022 - 0,1021 = 0,222 \quad (3).$$

But if $A < B$ a similar rule for subtraction (proceeding from left to right and borrowing when necessary unity from a later digit) would lead to an infinite sequence of digits. For example, if we take $p = 3$ and attempt to subtract, in this manner, 0,0022 from 0,1021, we obtain 0,10022..., in which the digit 2 is to be repeated indefinitely. It would be useless to define this symbol involving an infinitude of digits to be the infinite series

$$3 + 2 \cdot 3^4 + 2 \cdot 3^5 + \dots + 2 \cdot 3^n + \dots,$$

which is divergent and not equal to $138 - 216$. On the contrary we shall attach no numerical significance to such a symbol. Our procedure will be analogous to that employed in basing a theory of positive fractions upon the theory of positive integers by introducing symbols (a, b) involving a pair of integers. We here introduce symbols called p -adic numbers and define equality and the four rational operations. We shall prove that our set of p -adic numbers is closed under these operations and hence forms a field or domain. A certain subset of these p -adic numbers can be put into one-to-one correspondence with the rational numbers such that the sum, difference, product, or quotient of two p -adic numbers corresponds to the sum, etc., of the corresponding rational numbers. We shall then have a representation of each rational number as a p -adic number.

We note that, for $p = 3$, the quotient of $1 + 2 \cdot 3 + 3^2$ by 3 is $3^{-1} + 2 + 3$, which is conveniently denoted by the symbol $12,1 \quad (3)$.

Accordingly we introduce the symbols, called p -adic numbers,

$$D = d_{-p} d_{-p+1} \dots d_{-1} d_0 d_1 d_2 \dots \quad (p),$$

in which there is a finite number of *coefficients* d_i preceding the comma and a finite number or an infinitude of coefficients following the comma, while each d_i is a rational number in whose expression as a fraction in its lowest terms the denominator is not divisible by p . Such a fraction a/b is called *integral modulo* p since it plays the same rôle modulo p as the unique inte-

gral root of the congruence $bx \equiv a \pmod{p}$. A *reduced* p -adic number is one all of whose coefficients are integers of the set $0, 1, \dots, p-1$.

For $k \geq -\rho$, the rational number

$$D_k = d_{-\rho} p^{-\rho} + \dots + d_{-1} p^{-1} + d_0 + d_1 p + \dots + d_k p^k$$

is called the *convergent* D_k of rank k of the p -adic number D . For $k < -\rho$, we set $D_k = 0$. Thus if $D = 12, 1 \cdot (3)$, $D_{-1} = 3^{-1}$, $D_0 = 3^{-1} + 2$.

Two rational numbers will be called congruent modulo p^m if their difference equals the product of p^m by a number which is integral modulo p . Here m may be zero or any positive or negative integer. For example,

$$4 \cdot 5^{-2} \equiv \frac{2}{3} \cdot 5^{-2} \pmod{5^{-1}}, \quad 9 \cdot 5^{-2} \equiv \frac{2}{3} \cdot 5^{-2} \pmod{5^0},$$

since

$$4 \cdot 5^{-2} - \frac{2}{3} \cdot 5^{-2} = \frac{2}{3} \cdot 5^{-1}, \quad 9 \cdot 5^{-2} - \frac{2}{3} \cdot 5^{-2} = \frac{1}{3},$$

• while $\frac{2}{3}$ and $\frac{1}{3}$ are integral modulo 5.

Two p -adic numbers D and D' are said to be *equal* when for every integer k their convergents D_k and D'_k of rank k are congruent modulo p^{k+1} . Note that $D_k \equiv D'_k \pmod{p^{k+1}}$ implies $D_l \equiv D'_l \pmod{p^{l+1}}$ for every $l < k$. For example,

$$D = p, p-1, p-1, p-1, \dots, \quad D' = 0$$

are equal since

$$D_k = p + (p-1)p + \dots + (p-1)p^k = p^{k+1}, \quad D'_k = 0.$$

In particular, two reduced p -adic numbers are equal if and only if their corresponding coefficients are identical. Every p -adic number equals a reduced p -adic number. The proof follows from the fact that any rational number which is integral modulo p can be expressed in the form $i + lp$, where i is one of the integers $0, 1, \dots, p-1$, and l is integral modulo p . For example,

$$1\frac{2}{3}, 0 = 14, 1313 \dots (5)$$

with the repetend 13, since

$$\frac{2}{3} = 4 + (-\frac{2}{3})5, \quad -\frac{2}{3} = 1 + (-\frac{1}{3})5, \quad -\frac{1}{3} = 3 + (-\frac{2}{3})5.$$

If we prefix one or more zeros before a p -adic number D , we

obtain a p -adic number equal to D . Given any two p -adic numbers D and D' , we may prefix enough zeros before one so that we obtain two numbers with the same number ρ of coefficients to the left of the comma. Then the *sum* of D and D' is defined to be the p -adic number

$$D + D' = d_{-\rho} + d'_{-\rho} \dots d_{-1} + d'_{-1} \quad d_0 + d'_0, \quad d_1 + d'_1 \dots (p).$$

For example, $10,12 + 0,211 = 10,012$ (3). If $D = D_1$ and $D' = D'_1$, then $D + D'$ equals $D_1 + D'_1$.

The unique p -adic number X for which $X + D' = D$ is called the *difference* $D - D'$. Hence

$$D - D' = d_{-\rho} - d'_{-\rho} \dots d_{-1} - d'_{-1} \quad d_0 - d'_0, \quad d_1 - d'_1 \dots (p).$$

For example, employing a bar to denote a repetend, we have

$$0,\overline{12} - 0,\overline{210} = 0,\overline{201002} \quad (3).$$

The product P of two p -adic numbers D and D' is defined to be the p -adic number whose coefficients are those in the series obtained by the formal multiplication of the series

$$d_{-\rho}p^{-\rho} + \dots + d_{-1}p^{-1} + d_0 + d_1p + \dots,$$

$$d'_{-\rho'}p^{-\rho'} + \dots + d'_{-1}p^{-1} + d'_0 + d'_1p + \dots.$$

In particular, if D and D' are *integral* p -adic numbers,

$$D = d_0, d_1d_2 \dots, \quad D' = d'_0, d'_1d'_2 \dots (p),$$

their product is

$$P = d_0d'_0, \quad d_0d'_1 + d_1d'_0 \quad d_0d'_2 + d_1d'_1 + d_2d'_0 \dots (p).$$

In general, $P = c_{-\gamma}c_{-\gamma+1} \dots$, where $\gamma = \rho + \rho'$ and

$$c_{-\gamma} = d_{-\rho}d'_{-\rho'}, \quad c_{-\gamma+1} = d_{-\rho}d'_{-\rho'+1} + d_{-\rho+1}d'_{-\rho'}, \quad \dots$$

If the c 's and d 's are given numbers integral modulo p and d_ρ is not a multiple of p , the preceding equations uniquely determine each d' as a number integral modulo p , so that the quotient P/D is uniquely determined as a p -adic number $D' = d'_{-\gamma} \dots$. In particular, if D is a reduced p -adic number other than zero, P/D equals a p -adic number. For example,

$$3,12 \div 4,21 = 2,\overline{4220} \quad (5).$$

If $D = D_1$ and $D' = D'_1$, then $DD' = D_1D'_1$ and $D/D' = D_1/D'_1$.

For addition and multiplication as defined above the commutative, associative and distributive laws are seen to hold. The totality of p -adic numbers forms a field or Körper $K(p)$, being closed under the above defined operations addition, subtraction, multiplication, and division (the divisor not being equal to zero).

Any rational number $\neq 0$ may be given the form

$$r = \frac{a}{b} \cdot p^n,$$

where n is zero or a positive or negative integer, while a and b are integers not divisible by p . To r we make correspond the monomial p -adic number all but one of whose coefficients are zero, that one being a/b and occupying the n -th place to the right of the comma if n is positive, and the $(1 - n)$ -th place to the left of the comma if n is zero or negative. To the rational number zero we make correspond the p -adic number zero. The product or quotient of two rational numbers r and r' obviously corresponds to the product or quotient of the corresponding monomial p -adic numbers. A like result is seen to hold for the sum or difference, if we make use of the fact that the monomial p -adic number with the coefficient $p^k d$ in the n -th place equals a monomial with the coefficient d in the $(k+n)$ -th place. These monomial p -adic numbers when expressed as reduced p -adic numbers are periodic, and conversely any periodic p -adic number equals a monomial (Hensel, page 38). Hence the set of all periodic reduced p -adic numbers is simply isomorphic with the domain of all rational numbers. The p -adic number, whether reduced or not, which corresponds to the rational number r will be designated $[r]$.

Any p -adic number $E = e_0, e_1 e_2 \dots$, in which e_0 is not a multiple of p , is called a *unit* for the domain $K(p)$. Hence any p -adic number D can be expressed as a product $[p^n]E$. The exponent n is zero or a positive or negative integer and is called the *order* of D . The product or quotient of two units is a unit. The order of a product is the sum of the orders of the factors. A p -adic number $D = [p^n]E$ is called integral if its order n is positive or zero, and fractional if n is negative.

Consider an integral function of a variable x

$$f(x) = A_0 x^n + A_1 x^{n-1} + \dots + A_n$$

with p -adic coefficients A_i . If $A_i^{(k)}$ is the convergent of rank k of A_i , the function with rational coefficients

$$f^{(k)}(x) = A_0^{(k)}x^n + \dots + A_n^{(k)}$$

is called the convergent of rank k of $f(x)$. Two integral functions $f(x)$ and $g(x)$ with p -adic coefficients are called equal if, for every integer k , their convergents $f^{(k)}(x)$ and $g^{(k)}(x)$ are congruent modulo p^{k+1} , namely, if the coefficients of like powers of x are congruent.

An integral function is called reducible or irreducible in the domain $K(p)$ according as it is or is not equal to the product of two integral functions, each of degree ≥ 1 , with p -adic coefficients.

Any integral function $f(x)$ can be expressed as a product of a p -adic number and a *primary* function

$$F(x) = [p^a]x^n + B_1x^{n-1} + \dots + B_n$$

whose coefficients are integral p -adic numbers not all divisible by p , that of the highest power of x corresponding to a power of p . The product of two primary functions is primary. It is readily seen that $f(x)$ is reducible if and only if its primary component $F(x)$ is the product of two primary functions.

In case the discriminant $D(F)$ of $F(x)$ is zero, $F(x)$ and its derivative $F'(x)$ have a common factor which can be determined by Euclid's process. We may therefore restrict our attention to a primary function $F(x)$ whose discriminant is not zero and hence is of the form $[p^\delta]E$, where $\delta \geq 0$. Then (Hensel, page 68) $F(x)$ is reducible if and only if its convergent $F^{(k)}(x)$ is reducible modulo p^{k+1} . A similar argument (page 71) shows that if

$$F(x) \equiv f(x)g(x) \pmod{p^{r+1}}$$

and $r+1 > 2\rho$, where ρ is the order of the p -adic number defined by the resultant of $f(x)$ and $g(x)$, then $F(x)$ is reducible in $K(p)$. An important special case is the following. If

$$F(x) \equiv f(x)g(x) \pmod{p},$$

where f and g have no common factor modulo p , then $F(x)$ is reducible in $K(p)$. Since

$$x^{p-1} - 1 \equiv (x-1)(x-2)\dots(x-\overline{p-1}) \pmod{p},$$

we conclude that $x^{p-1} - 1$ has $p - 1$ linear factors in $K(p)$, so that there exist $p - 1$ p -adic numbers which are $(p - 1)$ -th roots of unity. They are all powers of a primitive $(p - 1)$ -th root of unity, designated by ω . For example,

$$\omega = 2, 22 \dots (3), \quad \omega = 3, 46 \dots (7).$$

Any p -adic number can be expressed in the form

$$B = p^\rho \omega^\beta e,$$

where e is a principal unit $1, a_1 a_2 \dots (p)$. If μ is not divisible by p , B is the μ -th power of a p -adic number if and only if ρ is divisible by μ and β is divisible by the greatest common divisor d of μ and $p - 1$ (page 87). In particular if p is odd, $\sqrt[p]{B}$ is a p -adic number only when ρ and β are both even. For example, $\sqrt[3]{2}$ is not a 3-adic number, since $2 = \omega e$; while $\sqrt[7]{2}$ is a 7-adic number, since $2 = \omega^2 e$. For p an odd prime, B is the p -th power of a p -adic number if and only if ρ is divisible by p and e is of the form $1, 0e_2 e_3 \dots (p)$.

A number β is called algebraic if it is the root of at least one equation

$$(1) \quad x^m + B_1 x^{m-1} + \dots + B_m = 0$$

with rational coefficients. If $B_1, \dots, B_m$ are integers, the root β is called an integral algebraic number. If $B_1, \dots, B_m$ are rational numbers which are integral modulo p , β is called an algebraic number integral modulo p . If β and γ are algebraic numbers integral modulo p then $\beta + \gamma, \beta - \gamma$ and $\beta\gamma$ are algebraic numbers integral modulo p . The roots of an equation

$$x^r + \beta_1 x^{r-1} + \dots + \beta_r = 0,$$

whose coefficients are algebraic numbers integral modulo p , are algebraic numbers integral modulo p .

If in (1) each $B_i = b_i/p^\rho$ where b_i is integral modulo p , then $\gamma = p^\rho \beta$ is a root of

$$\gamma^m + b_1 \gamma^{m-1} + p^\rho b_2 \gamma^{m-2} + \dots + p^{\rho(m-1)} b_m = 0.$$

Hence every algebraic number β can be given the form γ/p^ρ where γ is an algebraic number integral modulo p , and ρ is an integer ≥ 0 .

Let α be an algebraic number and let

$$(2) \quad F(x) = x^\lambda + a_1 x^{\lambda-1} + \dots + a_\lambda = 0$$

be the equation irreducible in the domain of rational numbers which has the root α . The totality of the rational functions of α with rational coefficients forms a field $K(\alpha)$ of degree λ . As shown in the theory of algebraic numbers, there exist λ integral algebraic numbers $\gamma_1, \dots, \gamma_\lambda$ of $K(\alpha)$ such that every algebraic number γ of $K(\alpha)$ can be expressed in one and but one way in the form

$$(3) \quad \gamma = u_1 \gamma_1 + u_2 \gamma_2 + \dots + u_\lambda \gamma_\lambda,$$

where each u_i is a rational number, while every integral algebraic number is of the form (3), where now each u_i is an integer. The numbers $\gamma_1, \dots, \gamma_\lambda$ are said to form a fundamental system for $K(\alpha)$. Since $\beta + \gamma$ and $\beta\gamma$ are algebraic numbers integral modulo p when β and γ are, it follows that

$$(4) \quad v_1 \gamma_1 + \dots + v_\lambda \gamma_\lambda$$

is an algebraic number integral modulo p if $v_1, \dots, v_\lambda$ are rational numbers integral modulo p . Conversely (Hensel, page 121), any algebraic number integral modulo p can be given the form (4).

An algebraic number β is said to be divisible by p^ρ if $\beta = p^\rho \gamma$, where γ is integral modulo p . If $\beta = b_1 \gamma_1 + \dots$, the conditions are $b_i \equiv 0 \pmod{p^\rho}$, for $i = 1, \dots, \lambda$.

Two algebraic numbers are called congruent modulo p^ρ if $\beta - \beta'$ is divisible by p^ρ . The conditions are $b_i \equiv b'_i \pmod{p^\rho}$ for $i = 1, \dots, \lambda$.

If β is integral modulo p , so that each b_i is a rational number integral modulo p , we can determine integers e_i of the set $0, 1, \dots, p-1$ such that $b_i = e_i + p c_i$, where c_i is integral modulo p . Hence $\beta = \epsilon + p\gamma$, where

$$\epsilon = e_1 \gamma_1 + \dots + e_\lambda \gamma_\lambda, \quad \gamma = c_1 \gamma_1 + \dots + c_\lambda \gamma_\lambda.$$

A number of the type ϵ is called a reduced number of $K(\alpha)$. Hence any algebraic number integral modulo p is congruent modulo p to a reduced number.

Just as we introduced a field $K(p)$ of all p -adic numbers containing a sub-field simply isomorphic with the field of all rational numbers, so we shall now introduce a field containing

a sub-field simply isomorphic with $K(\alpha)$. To this end we employ the symbols

$$\delta = \delta_{-p} \dots \delta_{-1} \delta_0, \delta_1 \delta_2 \dots (p),$$

called p -adic algebraic numbers, in which each δ_i is an algebraic number of $K(\alpha)$ integral modulo p . When each δ_i is a reduced number ϵ_i , δ is called a reduced p -adic algebraic number. By the convergent of rank μ of δ is meant

$$\delta^{(\mu)} = \delta_{-p} p^{-p} + \dots + \delta_{-1} p^{-1} + \delta_0 + \delta_1 p + \dots + \delta_{\mu} p^{\mu},$$

which is a number of the domain $K(\alpha)$. Two p -adic algebraic numbers δ and γ are called equal if, for every integer μ , their convergents of rank μ are congruent modulo $p^{\mu+1}$. In particular, if δ and γ are reduced, they are equal only when corresponding coefficients δ_i and γ_i are identical. In view of the preceding paragraph, any p -adic algebraic number β equals a reduced p -adic algebraic number.

Addition and multiplication of p -adic algebraic numbers is defined as for p -adic numbers. The totality of p -adic algebraic numbers based upon $K(\alpha)$ is seen to form a field $K(p, \alpha)$.

Any number $\beta \neq 0$ of $K(\alpha)$ can be expressed in one and but one way in the form $\beta = \gamma p^n$, where γ is a number of $K(\alpha)$ integral modulo p and not divisible by p , while n is zero or a positive or negative integer. To β we make correspond the monomial p -adic algebraic number all but one of whose coefficients are zero, that one being γ and occupying the n th place to the right of the comma if n is positive, and the $(1 - n)$ -th place to the left if n is zero or negative. When this monomial is expressed as a reduced p -adic algebraic number its coefficients $\epsilon_n, \epsilon_{n+1}, \dots$, form a periodic series. This follows from the fact that in (3) each u_i is rational and hence is represented by a periodic p -adic number. Hence $K(\alpha)$ is simply isomorphic to the sub-field of $K(p, \alpha)$ composed of the periodic reduced p -adic algebraic numbers. The p -adic algebraic number, whether reduced or not, which corresponds to the number β of $K(\alpha)$ will be designated $[\beta]$.

In the p -adic algebraic number δ , each coefficient δ_i is a linear function of $\gamma_1, \dots, \gamma_{\lambda}$. Hence the convergent $\delta^{(\mu)}$ equals $u_1^{(\mu)} \gamma_1 + \dots + u_{\lambda}^{(\mu)} \gamma_{\lambda}$, in which $u_i^{(\mu)}$ is the convergent of order μ of a p -adic number u_i . Hence

$$(5) \quad \delta = u_1[\gamma_1] + \dots + u_{\lambda}[\gamma_{\lambda}].$$

Let the function (2) with the root α be expressed as a product

$$(6) \quad F(x) = F_1(x) F_2(x) \cdots F_i(x) \quad (p)$$

in which $F_i(x)$ has p -adic coefficients, is irreducible in $K(p)$ and is of degree λ_i . Let $\alpha_1 = [\alpha]$, $\alpha_2, \dots, \alpha_{\lambda_i}$ be the roots of $F_i(x) = 0$.

Since the number γ_i of $K(\alpha)$ equals a rational function of α with rational coefficients, it follows from (5) that

$$\delta = \phi(\alpha_i) \quad (p),$$

where ϕ is a rational function with p -adic coefficients. Thus

$$\delta_j = \phi(\alpha_j) \quad (j = 1, \dots, \lambda_i)$$

are the roots of the equation

$$(7) \quad g_i(y) = \pi(y - \delta_j) = y^{\lambda_i} + b_1 y^{\lambda_i-1} + \dots + b_{\lambda_i} = 0 \quad (p),$$

with p -adic coefficients. If $g_i(y)$ has a factor $f(y)$ irreducible in $K(p)$ which vanishes for $y = \delta_j$, then $f[\phi(x)] = 0$ has one root α_i in common with the equation $F_i(x) = 0$ irreducible in $K(p)$ and hence has all the roots of the latter, so that $f(y)$ vanishes for each δ_j . Hence the various irreducible factors of $g_i(y)$ have the same roots and are therefore identical. Thus $g_i(y)$ is either irreducible or a power of an irreducible function. Multiplying $g_i(y)$ by a suitable power of p , we obtain

$$(7') \quad G_i(y) = B_0 y^{\lambda_i} + B_1 y^{\lambda_i-1} + \dots + B_{\lambda_i} = 0 \quad (p),$$

where the B_i are integral p -adic numbers not all divisible by p . It follows (Hensel, pages 74, 75) that B_0 and B_{λ_i} are not both divisible by p . Hence either δ or $1/\delta$ satisfies an equation with integral p -adic coefficients. A root of such an equation is said to be algebraically integral. A p -adic algebraic number ϵ is called an algebraic unit if both ϵ and $1/\epsilon$ are algebraically integral.

The product $\delta_1 \dots \delta_{\lambda_i}$ is called the partial norm $n_i(\delta)$ of $\delta = \delta_1$ with respect to the factor $F_i(x)$. Since

$$n_i(\delta) = (-1)^{\lambda_i} B_{\lambda_i} / B_0$$

is an integral p -adic number only when B_0 is not divisible by p , we conclude that δ is algebraically integral if and only if $n_i(\delta)$ is an integral p -adic number. In particular, ϵ is an algebraic unit if and only if $n_i(\epsilon)$ and its reciprocal are integral

p -adic numbers, namely, if $n_1(\epsilon)$ is a p -adic unit E . The product or quotient of two algebraic units is therefore a unit.

If δ and β are p -adic algebraic numbers, such that δ/β is algebraically integral, δ is said to be divisible by β . The condition is that $n_1(\delta)/n_1(\beta)$ shall be an integral p -adic number and hence that the order of $n_1(\delta)$ shall be equal to or greater than the order of $n_1(\beta)$. In case these orders are equal, δ/β is an algebraic unit ϵ and δ and β are said to be equivalent.

The norms $n_1(\delta)$ of all algebraically integral numbers δ not units, are of the form $p^d E$, where $d > 0$. Let f_1 be the least integer d , and π_1 a number for which

$$(8) \quad n_1(\pi_1) = p^{f_1} E.$$

If π is another number whose norm has the order f_1 , then $\pi = \pi_1 \epsilon$ so that π and π_1 are equivalent. Every algebraically integral number δ not a unit is divisible by π_1 since $d \geq f_1$. In particular, π_1 has no divisor other than itself and the units ϵ . If the product of two algebraically integral numbers is divisible by π_1 , one of the factors is divisible by π_1 . For, if neither δ nor δ' is divisible by π_1 , each is a unit and hence $\delta\delta'$ is a unit and not divisible by π_1 . Hence π_1 has the characteristic properties of a prime number, and all the primes in $K(p, \alpha)$ are equivalent. Since $n_1(p) = p^{f_1}$, π_1 is a divisor of p .

If β is any number of $K(p, \alpha)$, $n_1(\beta) = p^b E$. Set

$$b = \rho_1 f_1 + f_0, \quad 0 \leq f_0 < f_1.$$

Then $\beta' = \beta/\pi_1^{\rho_1}$ has the norm $p^{b-\rho_1 f_1} E' = p^{f_0} E'$. Hence $f_0 = 0$, so that β' is a unit ϵ . Hence every number β of $K(p, \alpha)$ is of the form $\beta = \epsilon \pi_1^{\rho_1}$. Here ρ_1 is called the order of β ; it is the quotient of the order of $n_1(\beta)$ by f_1 . In particular, $p = \epsilon \pi_1^{f_1}$, where $\epsilon_1 = \lambda_1/f_1$.

Two numbers β and β' of $K(p, \alpha)$ are called congruent modulo π_1 if $\beta - \beta'$ is divisible by π_1 . Hence if β is integral there is a reduced number $\epsilon^{(0)}$ such that

$$\beta \equiv \epsilon^{(0)} \pmod{\pi_1}, \quad \beta = \epsilon^{(0)} + \pi_1 \beta^{(1)}.$$

Similarly, $\beta^{(1)} = \epsilon^{(1)} + \pi_1 \beta^{(2)}$, etc. Hence

$$\beta \equiv \epsilon^{(0)} + \epsilon^{(1)} \pi_1 + \epsilon^{(2)} \pi_1^2 + \dots + \epsilon^{(k)} \pi_1^k \pmod{\pi_1^{k+1}}.$$

Any number δ of $K(p, \alpha)$ is of the form $\pi_1^{\rho_1} \epsilon$. Taking $\beta = \epsilon$, we obtain the congruence

$$(9) \quad \delta \equiv \epsilon^{(\rho_1)} \pi_1^{\rho_1} + \epsilon^{(\rho_1+1)} \pi_1^{\rho_1+1} + \dots + \epsilon^{(k)} \pi_1^k \pmod{\pi_1^{k+1}}.$$

We have $\pi_1^{\alpha_1} = p\epsilon$ where ϵ is a unit of $K(p, \alpha)$. Thus

$$\pi_1 = p^{1/\alpha_1} \epsilon_1,$$

where $\epsilon_1 = \epsilon^{1/\alpha_1}$ is an algebraic unit not belonging in general to $K(p, \alpha)$. Extending our definition of equivalence, we write

$$(10) \quad \pi_1 \sim p^{1/\alpha_1}.$$

If we replace α_1 by another root α_i of $F_1(x) = 0$, π_1 is replaced by a prime equivalent to p^{1/α_i} and hence to π_1 . Hence in view of (9) we obtain symbolical developments of the conjugate numbers $\delta(\alpha_i)$ in power series in p^{1/α_i} . These are analogous to the power series in $(z - \alpha)^{1/\alpha_i}$ for the roots of an algebraic equation in the neighborhood of a branch point $z = \alpha$ of order α_i .

For another factor $F_i(x)$ in (6) we have primes

$$\pi_i \sim p^{1/\alpha_i},$$

where α_i is a divisor of the degree λ_i of F_i . In order to avoid speaking of different, but equivalent, prime numbers, we associate with the factor $F_i(x)$ a unique prime divisor p_i and say that a p -adic algebraic number δ is divisible by $p_i^{\rho_i}$ and by no higher power of p_i if $\delta = \epsilon \pi_i^{\rho_i}$, so that the development, analogous to (9), begins with $\pi_i^{\rho_i}$. Thus p has the distinct prime divisors $p_1, \dots, p_r$. Every number γ of $K(\alpha)$ is therefore divisible by definite powers of these prime divisors. Then $\gamma\gamma'$ and γ/γ' are divisible by exactly the powers $\rho_i + \rho'_i$ and $\rho_i - \rho'_i$ of p_i . A number γ is algebraically integral if and only if it contains no one of the prime divisors p_i to a negative power.

If $x_1, \dots, x_{\lambda_i}$ are the roots of $F_i(x) = 0$, then

$$\delta_j = \phi(x_j) \quad (j = 1, \dots, \lambda_i)$$

are the roots of $g_i(y) = 0$, an equation analogous to (7). Thus

$$g(y) = g_1(y) \cdots g_i(y)$$

is a function whose constant term c_n differs at most in sign from the complete norm $n(\delta)$ which equals the product $n_1(\delta) \cdots n_i(\delta)$ of the partial norms of δ . Let now δ be an integral number of $K(\alpha)$. Then the coefficients of $g(y)$ are rational integers since its roots are $\phi(\beta)$ where β ranges over all the roots of (2). The highest power of p dividing $n(\delta)$ is p^d , where $d = \rho_1 f_1 + \cdots + \rho_r f_r$. But an integer c_n has only a finite number of prime factors. Hence an integral algebraic number γ has only a finite number of prime divisors.

In terms of these prime divisors, Hensel obtains (pages 214–237) an expression for the discriminant of $K(\alpha)$, that is, the discriminant of any fundamental system. Further he determines (pages 261–280) the conditions under which the discriminants of the various numbers of $K(\alpha)$ have a common unessential divisor. Hensel here makes an important advance in the theory of algebraic numbers. For other important results the reader is referred to the text itself.

In a book of the original character of the present one, some minor defects may be expected and excused. On page 16, A/B need not be, as stated, one of the integral p -adic numbers $c_0, c_1, \dots$. At the middle of page 34 occurs an equation, although equality of two fractional p -adic numbers has not yet been defined. If A, B, C are integral or fractional p -adic numbers such that $A = BC$, and if A_k, B_k, C_k are their convergents of rank k , it is stated on page 36 that

$$B_k \cdot C_k \equiv A_k \pmod{p^{k+1}}.$$

While this is true for integral p -adic numbers, it is in general false for fractional p -adic numbers. For if B is of negative order $-b$, and C of order $-c$, and $b \geq c$, we must employ the convergent of rank $k+b$ of C in order to reach all the terms of the convergent of rank k of A . The above congruence holds for the modulus p^{k+1-b} . On page 69, fourth line below (7), greater than p^p should be equal to or greater than p^p ; the proof is however valid. On page 96, before (2), read among all equations with rational coefficients. On page 121, fifth line from bottom, $v_i - v_i^{(0)}$ should have the denominator p . On page 123, the context shows the meaning of the term algebraically divisible; in the paragraph following (2), ganze should be preceded by modulus p , while $\bar{u}_i = p^b \bar{u}_i$ contains a misprint. In the eleventh line on page 124, the final letter p should be $\gamma^{(\lambda)}$. In the fifth line on page 135, B_λ should be B_0 . In the theorem on page 161, the term Bereich occurs in two senses; in the first instance it should be $K(p, \alpha)$, in the second, $K(p)$. On page 326, negative powers of p may occur in the p -adic development of B .

In the above exposition of the elements of Hensel's theory, I have avoided Hensel's notation $\Sigma_{i=-\infty}^{\infty} c_i p^i$ for a p -adic number, and have not identified the rational numbers with their corresponding p -adic representations. The terms greater than and less than as applied to p -adic numbers (Hensel, page 19) are not

in accord with the usage for real numbers, so that if the rational numbers a, b correspond to the p -adic numbers $[a], [b]$, we may have $a > b$, $[a] < [b]$. In setting up this correspondence, I have introduced the term monomial p -adic number. On page 130, Hensel assumes that the equation for α is irreducible in $K(p)$. Although not stated explicitly, this assumption underlies §§ 3–7 of the same chapter. In the present account I have therefore avoided this assumption and proceeded at once with the general case; see (6) above.

In addition to the intrinsic interest attached to the new fields or domains introduced by Hensel, his theory has proved to be of such importance in the difficult problems relating to discriminants that it must be granted a permanent footing in the theory of algebraic numbers.

L. E. DICKSON.

UNIVERSITY OF CHICAGO,
May 9, 1910.

SHORTER NOTICES.

Factor Table for the First Ten Millions. By D. N. LEHMER.
Washington, D. C., Carnegie Institution of Washington,
1909. xiv + 476 pp.

THE publication of Lehmer's factor table marks an event of the greatest importance in the science of higher arithmetic. The chief factor tables published hitherto are the following: For the first, second and third millions, Burckhardt (Paris, 1817, 1814, 1816); fourth, fifth and sixth millions, Glaisher (London, 1879, 1880, 1883); seventh, eighth and ninth millions, Dase and Rosenberg (Hamburg, 1862, 1863, 1865). Rosenberg's manuscript for the tenth million was presented by his widow to the Berlin Academy of Sciences, but has disappeared. Crelle's manuscript for the third, fourth, and fifth millions was turned over to the Berlin Academy but was found to be too inaccurate for publication. Kulik's manuscripts, placed in charge of the Vienna Royal Academy in 1867 (see *Encyklopädie der Mathematischen Wissenschaften*, volume I, page 951; *Wiener Berichte*, volume 53, page 460) purport to give the smallest factor of all numbers up to one hundred million which are not divisible by 2, 3, or 5. In Kulik's manuscript each prime not exceeding 163 is represented by a

single character (letter or digit), and the higher primes by two characters. Although his scheme effects a decided abbreviation, it increases greatly the difficulty of comparing his manuscript with other tables. Kulik's manuscript is not accurate enough for publication; his tenth million was translated into the ordinary notation by Lehmer and found to contain 226 erroneous entries. Lehmer gives a list of the 246 errors found in the tables by Burckhardt, Glaisher, Dase and Rosenberg for the first nine millions.

The publication of Lehmer's factor table is timely in view of the difficulty of obtaining copies of certain of the earlier tables. It is now practicable for each arithmetician to have at hand, ready for instant use, a factor table in a single volume extending to ten million.

However, the most important question in the comparison of two factor tables is their relative accuracy. When a computer uses a table giving the values of a continuous function, his result should be approximately correct; any considerable error may be laid at his own door. But when an arithmetician relies on a factor table for the primality of a given number, he is entirely dependent upon the accuracy of the table; the entry is either exactly right or wholly wrong, — there is no question of approximation. The independent verification that a proposed large number is actually prime usually entails great labor.

It is therefore in place to inquire into the grounds for the belief that Lehmer's factor table is more accurate than the earlier tables. The sheets (13 by 25 inches) of the corrected typewritten copy of the table were photographed on glass, reduced in size to 12 by 16 inches. Photographic proofs were then corrected by the author. The corrected photographs were transferred to zinc plates, on which any necessary corrections were made. It is believed that this reproduction by photography instead of by movable types has eliminated several sources of error in construction of the earlier tables. Mention should also be made of certain new devices employed in the actual construction of the present table, which tended towards increased accuracy. A certain modification enabled Lehmer to employ stencils only one-fourth of the length of Glaisher's. Consequently, Lehmer was able to employ the stencil method throughout, whereas Glaisher was compelled to have resort to the less accurate "multiple method" for primes higher than 307. Again, Lehmer's stencil possessed a certain symmetry

which was utilized to check its accuracy. Finally, Lehmer employed the improved plan of glueing the successive pages top to bottom in a continuous sheet, which was mounted upon a long bench with a roller at each end.

However, the claim for the greater accuracy of Lehmer's table does not rest so much upon these improvements in its construction as upon the fact that he was in a position to eliminate errors from the proof sheets by noting discrepancies with the earlier tables, as detected by a comparison entry for entry, a comparison made at least five times.

In Kulik's table the rate of error was approximately 1 in 1000 entries; in the other tables used for comparison the rate was lower. There is about an even chance that two computers working independently and with a rate as high as Kulik's will both make an error in the same place in a table extending to ten million. The probability that the same error will occur is therefore negligible.

An error in the case of a composite number is less likely to arise and is of less importance than in the case of a prime, as it would be detected at once by the person using the table. Hence the value of the check afforded by a count of the primes given in a table and its comparison with the number computed by Bertelsen (*Acta Mathematica*, volume 17) by a modification of Meissel's method (*Mathematische Annalen*, volumes 2, 21, 25). In a letter to the reviewer, Lehmer states that the actual count of the primes (including unity) in the first ten millions was found to be 664,580, which agrees with Bertelsen's computed number (increased by one to count unity), and that he has finished about three-fifths of a separate table of the primes less than ten million. In the latter work he checks each successive thousand with Glaisher's count and each successive fifty thousand with Bertelsen's computed number. No discrepancies with the latter have as yet appeared, while the frequent differences with Glaisher's count are all due to the presence of detected errors in the tables from which Glaisher made his count.

While the wide circle of mathematicians and amateurs interested in the theory of numbers is already under the greatest obligations to the Carnegie Institution of Washington for its aid to Lehmer in the construction of his factor table and for its publication, it will be fully satisfied only upon the completion of this monumental project by the publication of a separate table of primes.

L. E. DICKSON.

Analytische Geometrie des Punktpaares, des Kegelschnittes, und der Fläche zweiter Ordnung. Erster Teilband. By Dr. OTTO STAUDE. Leipzig and Berlin, Teubner, 1910. x + 548 pp., with 181 figures.

THIS volume is a continuation of the *Analytische Geometrie des Punktes, der geraden Linie und der Ebene* by the same author which was reviewed by Professor G. N. BAUER in the *BULLETIN*, volume 14, page 452 (June, 1908) and his concise summary applies equally well to the present volume. The treatment of the quadric surface is not completed, but is left for a third volume, to be devoted largely to intersections of surfaces of the second order. The bibliography and index will appear in the later volume.

The work aims to be a "Handbuch" and not simply "Vorlesungen." The author aims to leave no important facts concealed by attempting brevity, and as a result the reading of the book is apt to become tiresome because of detail. No effort is spared to make it convenient for ready reference. The table of contents, chapter heads, subheads, and important theorems are all so arranged as to make catch words stand out prominently. Important results are also frequently summed up in the form of tables, which are very convenient for comparison.

The book is divided into two parts of equal length. The first part deals with the point pair and the conic section. The second degree equation is treated in detail in cartesian, polar, parametric, and trilinear coordinates. Curves of the second class are treated as fully as those of the second order and line coordinates are used freely. Naturally there is little novel in this part. The classification of curves (as well as surfaces) according to "rank" is novel. A curve of the second order or class is of rank three if it has no double elements, of rank two if a single double element, and of rank one if a single infinity of double elements. Similarly an ordinary quadric surface is of rank four. This new use of the word rank does not seem necessary nor of particular advantage. The treatment of point and line involutions is of exceptional clearness. In treating class curves more use might be made of the principle of duality, and the Clebsch bordered determinant notation for the line equation of a conic could be used to advantage. The importance of the polar system with the conic as the locus of incident elements is properly emphasized. The section on confocal systems, introducing elliptic and parabolic

coordinates is also extremely well presented. In the chapter on trilinear coordinates the symbolic notation of Clebsch would be much briefer than the clumsy multiple summations, and fully as intelligible to most readers. Since the author writes down the simultaneous invariant of a conic and a line, it would seem worth while to point out that for variable coordinates u, v, w , this is merely the line equation of the conic.

It is somewhat disappointing that some of the features which make Salmon's Conic Sections a valued companion are not developed. There is nothing on invariants and covariants of systems of conics which are the subject of Salmon's most fruitful chapter. The theory of reciprocal polars is not very fully developed and such ideas as radius of curvature, evolutes, etc., are not introduced.

Part II. on surfaces is developed along similar lines. The most interesting features are probably the treatment of polar properties, the early introduction of line coordinates, and above all the clear discussion of the linear complexes arising. The next volume will doubtless contain much more of interest in space geometry.

D. D. LEIB.

Analytische Geometrie der Kegelschnitte. By W. DETTE. Leipzig, Teubner, 1909. vi + 232 pages, with 45 figures.

THIS admirable elementary text on conic sections is worthy of examination by any teacher of that subject. Both in arrangement and in treatment, there are a number of innovations. The book is divided into three parts: the first of 94 pages is devoted entirely to theory and the study of the general equation in the form $ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$; the next 50 pages contain a classified list of over twelve hundred examples illustrating the text; the remaining part of the volume is devoted to answers to, and also suggestions for solving, the examples of part two.

In the text itself, the six chapter headings: the point, the right line, the ellipse, the parabola, the hyperbola, and the determination of a conic through points and lines, promise little different from the old line American text. But on the first page the author introduces the idea of relative "mass numbers" or magnitudes; that is, for any segment of a line AB , we say $AB = -BA$. The author calls AB and BA the relative "mass numbers," their absolute magnitude being the same. As soon

as coordinates are defined, locus problems are solved and the equation of the circle derived before the straight line is mentioned. Oblique axes and polar coordinates are introduced from the start and every theorem or even remark which applies to oblique as well as rectangular coordinates is starred so as to attract attention. Transformations of coordinates are introduced only as needed. In common with most German texts the idea of harmonic points and lines is introduced early and kept in the foreground.

The fundamental division of curves represented by equations of the second degree is based on whether $ac - b^2 \gtrless 0$, although the equation of the ellipse is first derived from its focal property. In deriving the polar equation, p , the "parameter," is defined as the focal ordinate, hence taking the vertex as the origin we have the test $y^2 < 2px$ for the ellipse, $y^2 = 2px$ for the parabola, etc. The parabola is defined as an ellipse with one focus at infinity and all properties are derived from this definition, although it is later pointed out that it is equally well the limiting case of the hyperbola.

But perhaps the most striking feature in a book of such an elementary character is the extent to which the pole and polar idea is used. The entire theory of directrices, conjugate diameters, centers, tangents, and axes is based on this in a most attractive way. The "polarized" forms, i. e., $U_{11} = ax_1^2 + 2bx_1y_1 + cy_1^2 + 2dx_1 + 2ey_1 + f = 0$, $U_{12} = ax_1x_2 + \dots$, are regarded as fundamental quantities, and conditions are stated in terms of them; e. g., if $U_{11} \gtrless 0$, the point P_1 is outside, on, or within the curve; if $U_{12}^2 - U_{11}U_{22} = 0$, the line P_1P_2 is a tangent; if $U_{12} = 0$, the line P_1P_2 is harmonically divided by its intersections with the conic, and so on. While this is not new, it certainly is used more extensively than in any other elementary text.

The last chapter in the book is devoted largely to the theorems of Plücker, Pascal, and Brianchon. It must be very evident to the reader that the few pages of text contain most of the essentials of conic sections, presented in a most modern fashion. The author states that his only reasons for treating each of the conics separately are aesthetic and pedagogic.

D. D. LEIB.

Henri Poincaré. Biographie, Bibliographie analytique des Ecrits. Par ERNEST LEBON. Paris, Gauthier-Villars, 1909. viii + 80 pp.

THIS little volume is the first of a series which M. Lebon proposes to publish on the "Savants du Jour." The idea of giving to the public a biographical and bibliographical record concerning the various eminent scholars of the present day is novel and has much to recommend it. The reading of the present volume cannot but be an inspiration to the rising generation of mathematicians. The author, or more accurately the compiler (the book contains nothing from the pen of M. Lebon himself), has here brought together from various sources a series of articles and addresses relating to Poincaré. The contents are divided into seven sections relating respectively to biography, pure mathematics, analytic and celestial mechanics, mathematical physics, philosophy of science, necrology, and miscellaneous publications.

The section on Biography contains a part of the address delivered by M. Frédéric Masson, director of the Académie Française, at the time (January 28, 1909) when Poincaré was received as a member of the forty immortals. In fourteen pages we are given a comprehensive sketch of his life. Born at Nancy, April 29, 1854, he received his doctorate from the University of Paris in 1879. His rise to fame was meteoric. During the years 1879-1881 he published in the *Comptes Rendus* his epoch-making results on the uniformization of algebraic curves and the solution of linear differential equations which at the early age of 27 gave him a recognized position in the front rank of mathematicians. In 1885 he received from the Académie des Sciences the Prix Poncelet for the "ensemble de ses travaux mathématiques." The readers of the BULLETIN who wish to read in more detail of his life and work may be referred to a recent number of the *Popular Science Monthly* (September, 1909) which contains a translation of the address of M. Masson. This address is followed in the volume under review by a list covering four and a half pages of degrees, titles, prizes, membership in scientific bodies, etc.

The second Section is devoted to his work in pure mathematics. The text is an extract from the report by M. G. Rados on the Prix Bolyai (awarded to Poincaré by the Hungarian Academy of Sciences, April 18, 1905). This is followed by a list of publications in pure mathematics containing 146

titles. The text of the third Section is an extract from the address delivered by Professor G. H. Darwin (February 9, 1909) in presenting to Poincaré the gold medal of the Royal Astronomical Society of London. It is a brief review of his contributions to analytic and celestial mechanics. This is followed by a list of 85 titles of his publications in this field. The fourth Section opens with another extract from the report of M. Rados relating to Poincaré's contributions to mathematical physics. The list of his writings in this domain comprises 78 titles. The text of the fifth Section is a review by M. Emile Faguet of Poincaré's most recent book on the philosophy of science entitled "Science et Méthode." His writings in this field comprise 51 titles. His book "Science and Hypothesis" has been translated into five foreign languages. The Section devoted to necrology simply gives a list of 17 addresses and notices by Poincaré, on the life and work of Laguerre, Halphen, Tisserand, Weierstrass, Cornu, Berthelot, Kelvin, etc. The last section gives a list of 51 titles on miscellaneous subjects.

Aside from the inspiration to be derived from a reading of the text of this volume, the complete list of Poincaré's writings is valuable for reference. The volume also contains a fine heliogravure portrait of Poincaré.

J. W. YOUNG.

Des Notations Mathématiques, Énumération, Choix et Usage.
Par DÉSIRE ANDRÉ. Paris, Gauthier-Villars, 1909. xviii
+ 501 pp.

THE present work is practically confined, as regards both subject matter and method of treatment, to the needs of teachers of secondary mathematics. The paucity of historical references and the prolix treatment of many of the subjects tend to detract from its usefulness as a work of reference, and its general method of treatment seems better suited to the wants of those who are interested in seeing self-evident things stated in a clear and attractive form than of those who are seeking abstruse information. The perusal of such a work during periods of relaxation may, however, tend to make matters of secondary importance appear more attractive to the mathematician and thus it may lead to a keener appreciation of the entire mathematical structure.

The number of historical references is not only small but some of those which are given are apt to mislead the reader.

For instance, on pages 3 and 177 we are told that the Arabic number symbols were apparently derived from a set of broken lines or angles, equal in number to the units which the various symbols represent. In the first edition of Ball's *Short History of Mathematics*, page 147, this hypothesis as regards the origin of these symbols is presented, together with the following statement: "This conjecture is ingenious, but I am not aware of any historical basis for it." In the second edition of this history no reference seems to have been made to this "conjecture," and most of the other reliable histories pass it in silence. On the other hand no other theory as regards the origin of these universal symbols is mentioned in the book under review, and the unguarded reader would naturally infer that the theory here advanced had stronger historic support than any other.

As another instance, illustrating the author's tendency to pursue interesting conjectures rather than to be guided by tame historic facts, we may cite his treatment of the symbol $=$ on page 95. After stating that this symbol is attributed to Recorde, 1557, and after referring to the fact that it might have been selected to represent equality because of the equality of the two parallel line segments, he considers the probability of its having been derived from the ancient astronomical sign for the equinoxes. As is well known, Recorde stated explicitly that he selected this symbol because "noe 2 thynges can be moare equalle" than two equal and parallel line segments. It seems strange that any one should try to find better reasons for such a choice than those given by the man who made it.

From what has been said it results that the present work is not free from grave faults. On the other hand it has unusual merits. The lengthy discussion of the symbols in common use and the numerous references to details will doubtless lead many readers to observe matters which they formerly passed unnoticed. The book also offers mathematical reading matter which requires no more effort than the ordinary novel, and should be of no less interest to those who are familiar with the notation of elementary mathematics. It is divided into three parts, of nearly equal lengths, under the headings: enumeration, choice, and usage, respectively.

The first part is preceded by a preliminary discourse of fourteen pages, which closes with a statement of the object of the work and a brief description of the different parts. This part is devoted to an enumeration and explanation of the mathe-

mathematical symbols in actual use and it closes with directions as regards editorial work. The object is to give an exposition of the contemporaneous mathematical notation, omitting what is very seldom used. In the second and third parts a number of rules are given as regards the choice and the use of the symbols. Although these rules may appear to the reader as platitudes, yet their formulation is not without interest. The subject of mathematical notation is so important that such efforts to give a clear expression of what ought to be strict rules in the choice and use of notation deserve serious attention, especially on the part of the beginner.

The work closes with an unusually complete table of contents covering eleven pages. The number of the different subjects is very large. The various chapter headings are as follows: Nombres entiers, fractions, quantités déterminées, nombres indéterminés, signes d'opérations, signes de coordination, signes de fonctions, signes de relations, notations de la géométrie, signes de la géométrie analytique, mathématiques appliquées, signes de rédaction, netteté du signe, précision du signe, rappel des propriétés de l'objet, rappel des rapports entre les objets, choix des signes généraux, mesure des quantités, objets d'une seule sorte en nombre déterminé, objets d'une même sorte en nombre indéterminé, objets de deux sortes, correspondances entre deux sortes de signes, objets de plus de deux sortes, cas difficiles, écriture des expressions, expressions mal écrites, structure des expressions, expressions abrégées ou condensées, notations particulières, relations, systèmes d'équations, notations initiales des problèmes, mise en équations, direction des calculs, vérifications.

G. A. MILLER.

Applied Mechanics for Engineers. By E. F. HANCOCK. The Macmillan Company, 1909. xii + 385 pp.

IN writing this text the author has boldly, if not wisely, taken the viewpoint of the engineer in choosing the material which he considers essential, as well as in the method of presenting it. He has clearly in mind the fact that in an undergraduate course in mechanics for technical schools the greatest difficulty encountered by the student lies in seeing the application of theory to practice. As stated in the preface, no new material in the matter of principles and theory is presented. However, in the matter of application of the fundamental prin-

ciples the author has purposely entered the field of the engineer for new and up-to-date material. Of the 292 problems (with few answers) given for solution by the student many are adapted from original sources in engineering journals, and in this connection many references are cited. In this method of building up a text there lies the danger of having the choice of material influenced by local conditions, evidences of which crop out in many places in the book. It could easily be criticized by the engineer whose problems are mainly those of static equilibrium, and he would say that insufficient stress is placed on his line of work and that the side of engineering dealing with dynamics is over-emphasized. Certainly the strong influence of the latter is evident. Again, in the choice of material the text might tend to become an engineer's hand book, which at best would be insufficient in content and poorly arranged for reference. In this connection it might seem quite superfluous to include in appendices tables of logarithms, trigonometric functions, squares, roots, etc., which the student has used in a more convenient form long before he studies mechanics.

In bringing into the text practical engineering problems the author handles his material well, and in adapting this material the danger of leaving vague and sometimes erroneous notions concerning engineering projects in the minds of the students is reduced to a minimum. To help present these problems, and in order to give clear notions of the theory the text is illustrated with 205 well-drawn figures, each of which tells its story well because it holds fast to fundamentals. The print is clear and the typographical errors noted are few and unimportant. Approximations and empirical formulas are used frequently because these often arise in engineering practice, and comparisons of results obtained from their use with theoretical values are made.

If a table of contents were given here it would show that the text contains chapters on only those fundamental notions which all would agree should be found in a text on mechanics. The author's claim for the completeness of the chapters on moment of inertia, center of gravity, work and energy, friction, and impact, both from the standpoint of theory and application, may be granted; at least for the applications.

The book contains considerable engineering data of value, which would be increased materially if the students made a judicious use of the references cited, a few of the more involved

theoretical discussions, and a maximum of application. It is plainly a text for students in technical courses, while its use in a more general course would hardly be recommended.

ERNEST W. PONZER.

NOTES.

THE July number (volume 11, number 3) of the *Transactions of the American Mathematical Society* contains the following papers: "Die natürlichen Gleichungen der analytischen Curven in euclidischem Raume," by E. STUDY; "Two-dimensional chains and the associated collineations," by J. W. YOUNG; "Groups of rational transformations in a general field," by L. I. NEIKIRK; "On osculating element-bands associated with loci of surface-elements," by P. F. SMITH; "Fields of extremals in space," by G. A. BLISS, and M. MASON; "Groups generated by two operators s_1, s_2 satisfying the equation $s_1 s_2^2 = s_2 s_1^2$," by G. A. MILLER; "Congruences of the elliptic type," by L. P. EISENHART.

THE July number (volume 31, number 3) of the *American Journal of Mathematics* contains the following papers: "The osculants of plane rational quartic curves," by H. I. THOMSEN; "On the primitive groups of classes six and eight," by W. A. MANNING; "Minimalcurven als Oërter von Krümmungsmittelpunkten," and "Minimalcurven und Serret'sche Flächen," by E. STUDY; "On Steinerians of quartic surfaces," by J. N. VAN DER VRIES; "On the determination of the ternary modular groups," by R. L. BÖRGER; "Groups of transformations of Sylow subgroups," by G. A. MILLER.

THE concluding (July) number of volume 11 of the *Annals of Mathematics* contains: "The theory of shadow rails," by W. H. JACKSON; "On a new method of computing the roots of Bessel's functions," by W. MARSHALL; "A functional equation for the sine," by E. B. VAN VLECK; "Periodic decimal fractions," by W. H. JACKSON; "Concerning the invariant points of commutative collineations," by W. B. FITE; "A new construction for cycloids," by H. SCHAPPER; "Metric classification of conics and quadrics by means of rank," by G. RUTLEDGE; "A method of solving linear differential equations," by P. A. LAMBERT.

At the public Leibniz session of the royal Prussian academy of sciences held in Berlin on June 30 it was announced that as no competent memoir had been received, the subject announced in 1905 for which the Steiner prize was offered (*BULLETIN*, volume 12, page 42) is now withdrawn. The prize, of 6000 marks, was awarded to Professor GASTON DARBOUX in recognition of his contributions to the theory of surfaces.

The academy proposed the following problem for the solution of which the Steiner prize of 7000 marks will be awarded in 1914:

"To determine all those non-degenerate surfaces of order five on which one or more series of conics lie, and to investigate their properties. It is required to confirm the correctness and completeness of the solution by furnishing an analytic commentary on the results of the geometric investigation."

The academy also announced the following problem for the solution of which the academy's prize of 5000 marks will be awarded:

"The number of classes in the general cyclotomic field is to be determined and compared with the number of classes of its divisors."

In both cases competing memoirs should be written in German, French, Latin, English, or Italian, and be submitted to the secretary of the academy under the usual conditions on or before December 31, 1913.

THE royal academy of sciences of Belgium announces the following prize problems for the year 1911:

1. Develop and systematize our knowledge of the physical constitution of the sun. Prize of 800 francs.

2. Extend our knowledge of the nature of osmotic pressure. Prize of 600 francs.

3. A new contribution to the development of functions (real or analytic) in series of polynomials. Prize of 800 francs.

4. A resumé of the memoirs on systems of conics in space, together with a new contribution concerning such systems. Prize of 800 francs.

THE following courses in mathematics are announced for the winter-semester 1910-1911.

UNIVERSITY OF BERLIN. — By Professor H. A. SCHWARZ: Differential calculus, four hours; Applications of elliptic functions, four hours; Theory of complex numbers, two hours;

Colloquium, two hours ; Seminar, two hours. — By Professor G. FROBENIUS : Algebra, four hours ; Seminar, two hours. — By Professor F. SCHOTTKY : General theory of functions of a complex variable, four hours ; Theory of potential in the plane and in space, four hours ; Seminar, two hours. — By Professor G. HETTNER : Infinite series, products, and continued fractions, two hours. — By Professor J. KNOBLAUCH : Determinants, four hours ; Theory of surfaces and space curves, four hours ; Mathematical exercises, one hour. — By Professor R. LEHMANN-FILHÉS : Analytic geometry, four hours ; Planetary orbits, four hours.

UNIVERSITY OF BONN. — By Professor E. STUDY : Analytic mechanics, four hours ; Seminar, two hours. — By Professor F. LONDON : Analytic geometry of the plane and of space, four hours ; Descriptive geometry, three hours ; Seminar, two hours. — By Professor F. HAUSDORFF : Differential and integral calculus, II, four hours ; Introduction to the theory of groups, two hours. — By Dr. J. MÜLLER : Theory of potential, three hours.

UNIVERSITY OF GÖTTINGEN. — By Professor F. KLEIN : Development of mathematics in the nineteenth century, four hours ; Seminar, two hours. — By Professor D. HILBERT : Mechanics, four hours ; Seminar, two hours. — By Professor E. LANDAU : Differential and integral calculus, five hours ; Seminar, two hours. — By Professor C. RUNGE : Descriptive geometry, eight hours ; Seminar, two hours. — By Professor L. PRANDTL : Aerial mechanics, four hours ; Seminar, two hours. — By Dr. P. KOEBE : Determinants and applications, two hours ; Introduction to the theory of functions, six hours. — By Dr. O. TOEPLITZ : Algebra, four hours ; Theory of sets in elementary presentation, two hours. — By Dr. F. BERNSTEIN : Elliptic functions, four hours ; Calculus of insurance, two hours. — By Dr. C. MÜLLER : History of the discovery of the calculus, two hours. — By Dr. A. HAAR : Introduction to the calculus of variations, two hours ; Theory of perturbations, two hours. — By Dr. H. WEYL : Developments in series in mathematical physics, four hours.

UNIVERSITY OF LEIPZIG. — By Professor C. NEUMANN : Selected chapters of mathematics, three hours. — By Professor O. HOLDER : Analytic mechanics, five hours ; Theory of numbers, two hours ; Seminar, two hours. — By Professor K. ROHN :

Determinants, two hours ; Applications of the calculus to curves and surfaces, four hours ; Seminar, two hours. — By Professor G. HERGLOTZ : Differential and integral calculus, five hours ; Elements of the theory of partial differential equations, two hours ; Seminar, two hours. — By Professor H. LIEBMANN : Analytic geometry of space, five hours ; Non-euclidean geometry, two hours ; Seminar, two hours.

UNIVERSITY OF MUNICH. — By Professor F. LINDEMANN : Theory of functions of a complex variable, four hours ; Plane analytic geometry, four hours ; Introduction to the theory of groups of transformations, two hours ; Seminar, two hours. — By Professor R. v. SEELIGER : Celestial mechanics, four hours. — By Professor A. VOSS : Algebra, four hours ; Theory of differential equations, four hours ; Seminar, two hours. — By Professor A. PRINGSHEIM : Differential calculus, five hours ; Definite integrals and Fourier's series, four hours. — By Professor A. SOMMERFELD : Analytic mechanics, four hours ; Geometric optics, two hours. — By Professor L. BRUNN : Recent researches in analysis situs, two hours. — By Professor K. DOEHLEMAN : Descriptive geometry with exercises, eight hours ; Line geometry synthetically treated, four hours ; The imaginary in geometry, one hour. — By Dr. G. HARTOGS : Theory of space curves and surfaces, four hours.

UNIVERSITY OF STRASSBURG. — By Professor H. WEBER : Differential and integral calculus, four hours ; Elliptic functions, two hours ; Seminar, two hours. — By Professor F. SCHUR : Projective geometry of space, four hours ; Foundations of geometry, two hours ; Seminar, two hours. — By Professor J. WELLSTEIN : Introduction to the theory of algebraic functions, four hours ; Riemann surfaces, two hours ; Seminar, two hours. — By Professor L. v. MISES : Analytic geometry, four hours ; Kinematics of rigid bodies, two hours ; Seminar, two hours. — By Professor M. STAMON : History of mathematics, two hours. — By Professor P. EPSTEIN : Introduction to the theory of linear differential equations, two hours.

THE following advanced courses in mathematics are offered at the Italian universities during the academic year 1910–1911. Courses in algebra, analytic geometry, projective and descriptive geometry, and elementary courses in the calculus are not included :

UNIVERSITY OF BOLOGNA.—By Professor C. ARZELA: Lebesgue's integrals; advanced mechanics, three hours.—By Professor L. DONATI: Electromagnetic field, principle of relativity, three hours.—By Professor L. PINCHERLE: Distributive operations, integral and differential equations (especially of the second order, in both complex and real field), three hours.

UNIVERSITY OF CATANIA.—By Professor M. DE FRANCHIS: Differential geometry with applications to non-euclidean geometry, four hours.—By Professor L. LAURICELLA: Mathematical theory of elasticity with applications, four hours.—By Professor G. PENNACCHIETTI: Principles of celestial mechanics, with applications, four hours.—By Professor C. SEVERINI: General theory of functions, four hours.

UNIVERSITY OF GENOA.—By Professor E. E. LEVI: Foundations of the theory of functions of a real variable; calculus of variations, three hours.—By Professor G. LORIA: Theory of groups of transformations, three hours.—By Professor O. TEDONE: Special problems on equilibrium and motion of elastic bodies, three hours.

UNIVERSITY OF NAPLES.—By Professor F. AMODEO: History of mathematics before 1200, three hours.—By Professor R. MARCOLONGO: Vector analysis and its applications to hydro-mechanics, elasticity, and electro-mechanics, three hours.—By Professor D. MONTESANO: Birational correspondences in space; geometry of straight lines and of conics as elements of space, four and one half hours.—By Professor E. PASCAL: Linear differential equations, and the theory of groups of transformations, three hours.—By Professor E. PINTO: Physical optics with special regard to the phenomena of diffraction, four and a half hours.—By Professor G. TORELLI: Analytic theory of numbers, four and a half hours.

UNIVERSITY OF PADUA.—By Professor F. D'ARCAIS: General theory of functions, elliptic functions, four hours.—By Professor U. CISOTTI: Mathematical theory of elasticity with technical applications, three hours.—By Professor A. FARARO: The teaching of mathematics in the University of Padua from the fourteenth to the seventeenth century, three hours.—By Professor P. GAZZANIGA: Theory of numbers, three hours.—By Professor T. LEVI-CIVITA: Statistical mechanics; kinetic theory of gases, four and one half hours.—

By Professor G. RICCI: Absolute differential calculus; harmonic and poli-harmonic functions; general methods in elasticity, four hours.—By Professor F. SEVERI: Theory of algebraic functions of two variables and of their integrals, four hours.—By Professor G. VERONESE: Foundations of geometry, four hours.

UNIVERSITY OF PALERMO.—By Professor G. BAGNERA: Partial differential equations of the second order, three hours.—By Professor M. GEBBIA: Vibrations of elastic solid media; applications to acoustics and optics, four and one half hours.—By Professor G. B. GUCCIA: General theory of algebraic curves and surfaces, four and one half hours.—By Professor A. VENTURI: Planetary motion about the sun and about the center of mass; forms of planets, four and one half hours.

UNIVERSITY OF PAVIA.—By Professor E. ALMANSI: Theory of heat, three hours.—By Professor L. BERZOLARI: Geometry on an algebraic curve, three hours.—By Professor F. GERBALDI: Elliptic functions, three hours.—By Professor G. VIVANTI: General theory of functions with application to integral functions, three hours.

UNIVERSITY OF PISA.—By Professor E. BERTINI: Fundamental properties of the geometry on an algebraic surface, three hours.—By Professor L. BIANCHI: Preliminaries on ordinary and partial differential equations; infinitesimal geometry of curves and surfaces, four and one half hours.—By Professor U. DINI: Functions of a complex variable, elliptic functions, four and one half hours.—By Professor G. A. MAGGI: Equations of dynamics, potential and harmonic functions, vector fields with applications, four and one half hours.—By Professor P. PIZZERTI: Interpolation and numerical integration; notions in spherical astronomy, figure of the planets, three hours.

UNIVERSITY OF ROME.—By Professor G. BISCONCINI: Differential geometry in its connection to mechanics, three hours.—By Professor G. CASTELNUOVO: Foundations of geometry, non-euclidean geometry, three hours.—By Professor L. ORLANDO: Analytical foundation of mathematical physics, three hours.—By Professor V. VOLTERRA: General types of differential equations of mathematical physics and of celestial mechanics, three hours.—By —————: Higher analysis, three hours.

UNIVERSITY OF TURIN. — By Professor T. BOGGIO: Integral equations and theory of potential, three hours. — By Professor G. SANNIA: Non-euclidean geometry, three hours. — By Professor C. SEGRE: Geometry of birational transformations of curves and surfaces, three hours. — By Professor C. SOMIGLIANA: Theory of potential with applications, three hours. — By — : Higher analysis, three hours.

THE following mathematicians have been elected corresponding members of the royal academy of sciences of Turin: Professor J. BOUSSINESQ, of the University of Paris, Professor E. CAVALLI, of the University of Naples, Professor V. CERULLI, of Teramo, Italy, Professor Sir G. H. DARWIN, of Cambridge University, Professor F. ENRIQUES, of the University of Bologna, Professor G. P. GUCCIA, of the University of Palermo, Professor T. LEVI-CIVITA, of the University of Padua, Professor C. NEUMANN, of the University of Leipzig. Professor M. NOETHER, of the University of Erlangen, has been elected foreign member.

At the University College of London, Mr. F. JACKSON and Miss M. PICK have been appointed associate professors of mathematics. Mr. P. F. EVERITT has been appointed honorary associate professor of mathematics.

DR. M. PLANCHEREL has been appointed docent in mathematics at the University of Geneva.

DR. L. GABEREL, of the University of Neuchâtel, has been promoted to an associate professorship of mathematical analysis.

DR. P. FURTWÄNGLER, of the technical school at Aachen, has been appointed professor of mathematics at the agricultural institute of Bonn-Poppelsdorf.

DR. L. LICHTENSTEIN has been appointed docent in mathematics at the technical school of Berlin.

DR. E. FISHER has been promoted to an associate professorship of mathematics at the technical school of Brünn.

PROFESSOR F. HASENÖHRL has been elected corresponding member of the academy of sciences of Vienna.

PROFESSOR B. HOPKINSON has been elected a member of the royal society of London.

PROFESSOR K. VON DER MÜHLL has received the honorary degree of doctor of laws from the University of Basel.

PROFESSOR H. A. SCHWARZ, of the University of Berlin, has been decorated with the order of the red eagle of the third class with shield.

DR. C. MÜLLER, of the University of Göttingen, has been appointed professor of mathematics at the technical school of Hannover.

PROFESSOR E. STEINITZ, of the technical school at Berlin, has been appointed professor of higher mathematics at the technical school of Breslau.

DR. TH. V. KÁRMÁN has been appointed docent in mechanics at the University of Göttingen.

PROFESSOR G. LAURICELLA, of the University of Catania, has been appointed professor of mathematics at the University of Rome, as successor to the late Professor CERRUTI.

PROFESSORS E. CIANI, of the University of Genoa, B. LEVI, of the University of Cagliari, and G. FUBINI, of the technical school of Turin, have been promoted to full professorships of mathematics.

DR. L. SILLA has been appointed docent in rational mechanics at the University of Rome.

DR. C. AIMONETTI has been appointed docent in geodesy at the University of Turin.

DR. G. FORNI has been appointed docent in geodesy at the technical school of Milan.

DR. W. A. GRANVILLE, of Yale University, has been elected president of Pennsylvania College, Gettysburg, Pa.

DR. R. P. BAKER, of the University of Iowa, has been promoted to an assistant professorship of mathematics.

MISS M. B. WHITE has been appointed assistant professor of mathematics at the University of Kansas. Mr. G. W. HESS has been appointed instructor in mathematics.

PROFESSOR OSWALD VEBLEN, of Princeton University, has been promoted to a full professorship of mathematics.

PROFESSORS J. B. SHAW, of the James Milliken University, and ARNOLD EMOH, of the cantonal school of Soloturn, Switzerland, have been appointed assistant professors of mathematics at the University of Illinois.

THE following appointments to instructorships in mathematics are announced: Dr. C. D. BROOKS, at Northwestern University; Mr. B. E. CARTER, at Colby College; Mr. E. E. MOOTS, at the University of Wisconsin; Mr. C. J. WEST, at the University of Ohio; Mr. R. S. POND, at the University of Georgia.

PROFESSOR J. WEINGARTEN, of the University of Freiburg, died June 16, at the age of 74 years.

PROFESSOR J. V. SCHIAPARELLI, former director of the astronomical observatory at Milan, died July 4, at the age of 75 years.

PROFESSOR L. RAFFY, of the University of Paris, died June 9, at the age of 55 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

EPSTEIN (P.). See PASCAL (E.).

FUCHS (C. A.) und LUDWIG (F. K.). Zur Geometrie zweier und mehrerer Kreise. Komotau, Benker, 1910. 8vo. 54 pp. M. 1.30

HIME (H. W. L.). Anharmonic coordinates. London, Longmans, 1910. 12mo. 7s. 6d.

LUDWIG (F. K.). See FUCHS (C. A.).

MONTELL (P. L.). Théorie du point (2e partie). Géométrie curviligne. 2e édition, revue, corrigée et considérablement simplifiée. Paris, Chapelot, 1910. 4to. 82 pp.

MULLER (E.). See SCHRÖDER (E.).

NETTO (E.). Die Determinanten. Leipzig, Teubner, 1910. 8vo. 6 + 129 pp. Cloth. M. 3.60

PASCAL (E.). Repertorium der höheren Mathematik. 2te völlig umgearbeitete Auflage der deutschen Ausgabe, herausgegeben von H. E. Timerding und P. Epstein. Band II, erste Hälfte: Grundlagen und ebene Geometrie. Leipzig, Teubner, 1910. 8vo. 16 + 534 pp. M. 10.00

SALMOIRAGHI (A.). Guida pratica della geometria moderna. Milano, Allegratti, 1910. 8vo. 24 + 120 pp. L. 5.00

SCHIMANSKI (E.). Die algebraischen Invarianten der projektiven Gruppen der Ebene und die geometrische Charakterisierung dieser Gruppen. (Diss.) Königsberg, 1910. 8vo. 95 pp.

- SCHRÖDER (E.). Abriss der Algebra der Logik. Bearbeitet im Auftrag der deutschen Mathematiker-Vereinigung. Von E. Müller. Teil II: Aussagentheorie, Funktionen, Gleichungen und Ungleichungen. Leipzig, Teubner, 1910. 8vo. 6 + 51-159 pp. M. 4.00
- SCHRÖDER (J.). Aufgaben für den Unterricht in der analytischen Geometrie der Ebene an höheren Schulen. Leipzig, Teubner, 1910. 8vo. 4 + 49 pp. Cloth. M. 1.40
- SIGNORINI (A.). La trasformazione Bk delle superficie applicabili sulle quadriche dello spazio ellittico. Pisa, Nistri, 1909. 8vo. 124 pp.
- SONNET (H.) y FRONTERA (G.). Elementos de geometría analítica. Madrid, Bailly-Baillière, 1910. 8vo. 682 pp. P. 8.00
- STOCKHAUS (H.). Beitrag zum Beweis des Fermatschen Satzes. Leipzig, Brandstetter, 1910. 8vo. 7 + 90 pp. M. 2.00
- TIMMERDING (H. E.). See PASCAL (E.).

II. ELEMENTARY MATHEMATICS.

- BELLACCHI (G.). Complementi di geometria ed'algebra. Firenze, Barbèra, 1910. 16mo. 161 pp. L. 3.00
- BERRANO (G. B.). Geometria per uso delle scuole tecniche. 2a edizione, corretta. Torino, Olivero, 1910. 8vo. 186 pp. L. 2.60
- BOUGUERET (A.). Cours de dessin géométrique à l'usage des enseignements secondaire, primaire et professionnel. 2me partie: Projections; Éléments; Rotation des-solides; Sections planes. Paris, Thomas, 1910. 4to. 10 pp.
- . Cours de dessin géométrique à l'usage des enseignements secondaire, primaire et professionnel. 6ème partie: Architecture. Paris, Thomas, 1910. 4to. 10 pp.
- BRIGGS (W.). Second stage mathematics, with modern geometry. London, Clive, 1910. 12mo. 3s. 6d.
- D'OVIDIO (E.). See SANNIA (A.).
- DEUKES (J.). Ausführlicher Lehrgang der Arithmetik und Algebra nach modernen Grundsätzen. Für Lehrer und Studierende. Band I. Köln, M. du Mont-Schauberg, 1910. 8vo. 7 + 445 pp. M. 8.00
- DURELL (C. V.). A course in plane geometry for advanced students. Part II. New York, Macmillan, 1910. 8vo. 14 + 358 pp. Cloth. \$2.40
- EBNETER (K.). Aufgaben der elementaren Algebra. Methodisch geordnete Sammlung für die 3te Klasse an Sekundar- oder Realschulen, sowie für Gewerbeschulen und verwandte Lehranstalten. St. Gallen, Fehr, 1910. 8vo. 6 + 67 pp. M. 0.80
- FERNIER y PADILLA (F.). Nociones de aritmética y geometría. Barcelona, Carbonell, 1910. 214 pp. P. 2.00
- FURLE (H.). Ein Rechenblatt zur Auflösung der Gleichung vierten Grades mit Hilfe des Zirkels. (Progr.) Berlin, Weidmann, 1910. 8vo. 16 pp. M. 1.00

- GOLLER (J.). Raumlehre mit geometrischem Zeichnen, bearbeitet im Auftrage des württembergischen evangelischen Lehrerunterstützungsvereins. Schülerausgabe A, für die Volksschule. 2te Auflage. Stuttgart, Bonn, 1910. 8vo. 48 pp. M. 0.40
- HOLLMANN (T.). Mathematik für höhere Mädchenschulen. Nach den neuen Lehrplänen bearbeitet. II. Arithmetik und Algebra. Düsseldorf, Schwann, 1910. 8vo. 6 + 175 pp. Cloth. M. 2.50
Lösungen. 36 pp. M. 0.80
- JAQUET (E.) et LACLEF (A.). *Éléments d'algèbre. Exercices et problèmes à l'usage des écoles primaires supérieures, des écoles professionnelles, des écoles normales d'instituteurs.* Nouvelle édition. Paris, Nathan, 1910. 16mo. 190 pp.
- LAYNG (A. E.). *A general text-book of elementary Algebra.* Books 1 and 2. London, Blackie, 1910. 12mo. 2s. 6d.
- MARVÁ Y MEYER (J.). See RODRÍGUEZ PERLADO (P.).
- MOORE (E. H.). See MYERS (G. W.).
- MULLER (H.) und MAHLERT (A.). *Mathematisches Lehr- und Übungsbuch für höhere Mädchenschulen. Teil I: Arithmetik und Algebra.* 3te Auflage. Leipzig, Teubner, 1910. 8vo. 6 + 146 pp. Cloth M. 2.00
- MYERS (G. W.). *Second-year mathematics for secondary schools; edited by E. H. Moore.* Chicago, University of Chicago Press, 1910. 14 + 282 pp. Cloth. \$1.50
- NAMFON (G.). *Cent questions de théorie. Arithmétique, algèbre, géométrie des examens du brevet supérieur avec développements et solutions.* 2me édition, revue. Paris, Hachette, 1910. 16mo. 111 pp. Fr. 1.00
- ONBERG (E.). *Use of logarithms and logarithmic tables. With tables of logarithms.* New York, Industrial Press, 1910. 8vo. 35 pp. \$0.25
- RODRÍGUEZ PERLADO (P.). *Tratado de geometría elemental.* Con prólogo de J. Marvá y Meyer. Madrid, Hernando, 1910. 391 pp. P. 7.50
- SALAZAR é IBÁÑEZ (M.). *Problemas matemáticas. Contestaciones al programa de esta asignatura, para las oposiciones al Cuerpo de Aduanas.* Madrid, Marzo, 1910. 8vo. 300 pp. P. 7.00
- SANNIA (A.) e D'OVIDIO (E.). *Elementi di geometria. Vol. I, ad uso dei ginnasii e degli istituti tecnici.* 13a edizione, riveduta e corretta. Napoli, Pellerano, 1910. 16mo. 15 + 204 pp. L. 2.00
- SCHULKE (A.). *Aufgaben-Sammlung aus der reinen und angewandten Mathematik. 2ter Teil, für die oberen Klassen höheren Schulen.* 2te Auflage. Leipzig, Teubner, 1910. 8vo. 10 + 186 pp. Cloth. M. 2.40
- SCHULTZE (A.). *Elements of Algebra.* New York, Macmillan, 1910. 12mo. 12 + 309 pp. Cloth. \$0.85
- WEBER (H.) und WELLSTEIN (J.). *Encyklopädie der Elementar-Mathematik. Ein Handbuch für Lehrer und Studierende. Band I: Encyklopädie der elementaren Algebra und Analysis.* Bearbeitet von H. Weber. 3te Auflage. Leipzig, Teubner, 1910. 8vo. 18 + 532 pp. Cloth. M. 10.00
- WEBER (K.). *Lehrbuch der Planimetrie für höhere Schulen und zum Selbstunterricht.* Wolfenbittel, Zwissler, 1910. 8vo. 164 pp. M. 3.00

WELLSTEIN (J.). See WEBER (H.).

WINKLER. Lehrbuch der Geometrie für höhere Mädchenschulen. Auf Grund der ministeriellen Bestimmung über die Neuordnung des höheren Mädchenschulwesens vom 8. VIII und 12. XII. 1908 nach Spieker's "Lehrbuch der Geometrie für höhere Lehranstalten" bearbeitet. Teil 3, Klasse II der höheren Mädchenschule. Potsdam, Stein, 1910. 8vo. 36 pp. M. 0.50

III. APPLIED MATHEMATICS.

ALBE (E. E. F. d'). The electron theory; a popular introduction to the new theory of electricity and magnetism; with a preface by G. J. Stoney. 3rd edition. New York, Longmans, 1910. 12mo. 27 + 327 pp. Cloth. \$1.25

BAZARD (A.). Cours de mécanique. Vol. IV: Chaudières et machines à vapeur. Turbines à vapeur. Paris, Geisler, 1910. 8vo. 593 pp. Fr. 12.00

BEAUFILS (H.). See HARANG (F.).

BORCHARDT (W. G.). Key to elementary statics. London, Rivingtons, 1910. 12mo. 10s. 6d.

BURK (E.). See HESS (L.).

CAUNT (G. W.). See JESSOP (C. M.).

CHOLLET (T.) et MINEUR (P.). Traité de géométrie descriptive. 7e édition. Paris, Vuibert et Nony, 1910. 8vo. 6 + 336 pp.

DREYER (G.). Elemente der Graphostatik. Lehrbuch für technische Unterrichtsanstalten und zum Selbstunterricht, mit vielen Anwendungen auf den Maschinenbau und Brückenbau. 3te Auflage. Ilmenau, Reinmann, 1910. 8vo. 6 + 121 pp. Cloth. M. 7.50

DUNCAN (R. H.). Practical curve tracing. With chapters on differentiation and integration. New York, Longmans, 1910. 8vo. 8 + 138 pp. Cloth. \$1.60

FARJON. See GOUARD (E.).

FERVAL (H.). Eléments de géométrie descriptive à l'usage des candidats aux baccalauréats de l'enseignement secondaire et aux écoles du gouvernement. 10me édition, conforme aux programmes du 27 juillet 1905. Paris, Belin, 1910. 12mo. 350 pp. Fr. 3.75

FISCHER (M.). Statik und Festigkeitslehre. Vollständiger Lehrgang zum Selbststudium für Ingenieure, Techniker und Studierende. 1ter Band: Grundlagen der Statik und Berechnung vollwandiger Systeme einschliesslich Eisenbeton. 2te, vermehrte Auflage. Mit zahlreichen Beispielen und Zeichnungen. Berlin, Meusser, 1910. 8vo. 10 + 645 pp. Cloth. M. 18.00

FUCINI (C.). Lezioni di geometria descrittiva. Genova, Gioventù, 1910. 8vo. 80 pp. L. 1.80

GOUARD (E.) et HIERNAUX (G.). Cours élémentaire de mécanique industrielle. Principes généraux, applications, exercices pratiques. Préface de M. Farjon. A l'usage des écoles pratiques de commerce et d'industrie (rédigé conformément aux programmes du 28 août 1909), des écoles nationales professionnelles, des écoles des mécaniciens de la marine, etc. Vol. I. Paris, Dunod, 1910. 8vo. 16 + 328 pp. Fr. 4.00

- GRANDA Y CALLEJAS (B.). Curso de mecánica aplicada á las construcciones. Resistencia de materiales y estabilidad de las construcciones. Vols. I, II. Madrid, Valentín Tordesillas, 1910. 8vo. 1195 pp. P. 55.00
- GROSSMANN (L.). Fragmente neuerer mathematisch-technischer Disciplinen der Versicherungs- und Finanzwissenschaft, begründet auf Ergebnissen selbständiger, exact wissenschaftlicher Forschung; mit Commentaren und Ergänzungen zu dem Werke. "Die Mathematik im Dienste der Nationalökonomie." 5ter Teil. Wien, 1909. 8vo. 64 pp. M. 5.00
- HARANG (F.) et BEAUFILS (H.). Notions élémentaires de géométrie descriptive appliquée au dessin. A l'usage des écoles pratiques de commerce et d'industrie (rédigé conformément aux programmes du 28 août 1909). Paris, Dunod, 1910. 8vo. 12 + 172 pp. Fr. 2.50
- HAERPFER (A.). Die Probleme von Hansen und Snellius. (Abhandlungen zur Geschichte der mathematischen Wissenschaften, 26, I.) Leipzig, Teubner, 1910. 8vo. 20 pp. M. 1.00
- HESSE (L.). Baumechanik für Hoch- und Tiefbautechniker. Verfasst für Schule und Praxis. Herausgegeben von E. Burck. 2te, vermehrte und verbesserte Auflage. Wien, Fromme, 1910. 8vo. 10 + 274 pp. M. 6.50
- HIERNAX (G.). See GOUARD (E.).
- JAMESON (A.). A text-book of applied mechanics and mechanical engineering. Vol. 5. 7th edition, revised and enlarged. London, Griffin, 1910. 8vo. 522 pp. 7s. 6d.
- JESSOP (C. M.) and CAUNT (G. W.). The elements of hydrostatics. London, Bell, 1910. 8vo. 134 pp. Cloth. 2s. 6d.
- JUDE (A.). The theory of the steam turbine. 2d edition. London, Griffin, 1910. 8vo. 436 pp. 18s
- KELVIN (LORD). Mathematical and physical papers. Vol. 4. Hydrodynamics and general dynamics. Cambridge, University Press, 1910. 8vo. 580 pp. Cloth. 18s.
- LEROY (C. F. A.). Traité de géométrie descriptive, suivi de la méthode des plans cotés et de la théorie des engrenages cylindriques et coniques, avec une collection d'épures, composée de 71 planches. 15me édition, revue et annotée par E. Martelet. Vol. I: Texte. Vol. II: Planches. Paris, Gauthier-Villars, 1910. 4to. Vol. I. 20 + 369 pp. Vol. II. 71 pp. Fr. 16.00
- MAJERCEK (W.). Die Berechnung elektrischer Freileitungen nach wirtschaftlichen Gesichtspunkten. Berlin, Springer, 1910. 8vo. 59 pp. M. 2.00
- MARTELET (E.). See LEROY (C. F. A.).
- MAUDERLI (S.). Untersuchungen über Stabilität dynamischer Systeme in der Mechanik des Himmels. (Diss.) Zürich, 1910. 8vo. 152 pp.
- MAYMÓ (M.). Manual de mecánica aplicada. Barcelona, Marín, 1910. 8vo. 590 pp. P. 6.00
- MINEUR (P.). See CHOLLET (T.).
- RICHARDSON (L. P.). The approximate arithmetical solution by finite differences of physical problems involving differential equations, with an application to the stresses in a masonry dam. London, Dulau, 1910. 4to. 3s. 6d.

ROY (L.). Recherches sur les propriétés thermomécaniques des corps solides (thèse). Paris, Gauthier-Villars, 1910. 4to. 75 pp.

RUDOLPH (H.). Die mechanische Erklärung der Naturerscheinungen, insbesondere der Relativbewegung, des Planckschen Wirkungselements und der Gravitation. Coblenz, Gross, 1910. 8vo. 6 + 68 pp. M. 2.00

SALAZAR é IBÁÑEZ (M.). Mecánica. Contestaciones al programa de esta asignatura, para las oposiciones al Cuerpo de Aduanas. Madrid, Marzo, 1910. 8vo. 379 pp. P. 8.00

STONEJ (G. J.). See ALBE (E. E. F. D').

VOGLER (A.). Geodätische Uebungen für Landmesser und Ingenieure. 3te Auflage. Teil I: Feldübungen. Berlin, Parey, 1910. 8vo. 8 + 311 pp. Cloth. M. 10.00

WOODWARD (C. J.). A B C five-figure logarithms and tables for chemists. London, Simpkin, 1910. 12mo. 72 pp. 2s. 6d.

THE SEVENTEENTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE seventeenth summer meeting of the Society was held at Columbia University on Tuesday and Wednesday, September 6-7, 1910, two sessions being arranged for each day. The attendance included the following thirty-six members :

Professor M. J. Babb, Professor G. D. Birkhoff, Professor J. W. Bradshaw, Professor E. W. Brown, Dr. H. T. Burgess, Mr. R. D. Carmichael, Professor G. G. Chambers, Professor A. B. Coble, Professor F. N. Cole, Dr. G. M. Conwell, Professor John Eiesland, Professor T. S. Fiske, Professor W. B. Fite, Professor O. E. Glenn, Professor C. C. Grove, Professor H. E. Hawkes, Dr. A. A. Himowich, Professor J. I. Hutchinson, Professor O. D. Kellogg, Professor C. J. Keyser, Professor P. A. Lambert, Dr. D. D. Leib, Mr. Joseph Lipke, President E. O. Lovett, Dr. H. H. Mitchell, Professor Arthur Ranum, Professor D. A. Rothrock, Dr. J. E. Rowe, Professor H. E. Slaughter, Dr. Clara E. Smith, Professor Virgil Snyder, Professor R. P. Stephens, Dr. W. M. Strong, Professor Oswald Veblen, Professor H. S. White, Professor Alexander Ziwet.

Vice-President Professor Hutchinson occupied the chair. The Council announced the election of the following persons to membership in the Society : Mr. F. S. Bartlett, General Electric Company, Schenectady, N. Y. ; Mr. R. D. Beetle, Dartmouth College ; Professor N. C. Grimes, University of Arizona ; Professor F. T. H'Doubler, Miami University ; Mr. Robert Henderson, Equitable Life Assurance Society, New York, N. Y. ; Mr. G. F. McEwen, Stanford University ; Professor Josephine A. Robinson, Berea College. Ten applications for membership were received.

Professors P. F. Smith, Osgood, and E. H. Moore were appointed a committee to report to the Council at the October meeting a list of nominations for officers and other members of the Council to be elected at the annual meeting in December.

On Tuesday evening twenty-five members gathered at the usual dinner, which is always an agreeable incident of the meetings.

The following papers were read at the summer meeting:

(1) Professor L. E. DICKSON: "On the factorization of integral functions with p -adic coefficients."

(2) Professor L. E. DICKSON: "Determination of the binary modular groups and their invariants."

(3) Professor O. E. GLENN: "On the structure of p -ary forms."

(4) Mr. R. D. CARMICHAEL: "Linear difference equations and their analytic solutions."

(5) Dr. H. T. BURGESS: "The simultaneous reduction of a quadratic and a bilinear form by the same transformation on both x 's and y 's."

(6) Professor A. B. COBLE: "On the reduction of the sextic equation to the Valentiner form problem."

(7) Professor VIRGIL SNYDER: "The involutorial birational transformation of the plane, of order 17."

(8) Professor L. E. DICKSON: "An invariantive investigation of irreducible binary forms."

(9) Professor ARTHUR RANUM: "On the classification of systems of linear equations."

(10) Professor ARTHUR RANUM: "The osculating sphere of a developable surface."

(11) Professor PETER FIELD: "The theory of degenerate rational plane curves."

(12) Dr. J. E. ROWE: "Important covariant curves and a complete system of invariants of the rational quartic curve."

(13) Professor G. D. BIRKHOFF: "General theory of linear difference equations."

(14) Professor L. E. DICKSON: "A fundamental system of invariants of the general modular linear group on m variables."

(15) Professor S. E. SLOCUM: "A general formula for the shearing deflection of beams of arbitrary cross section, either variable or constant."

(16) Professor G. A. MILLER: "Note on the solution of a system of linear equations."

(17) Professor G. A. MILLER: "Some relations between substitution group properties and abstract groups."

(18) Professor JACOB WESTLUND: "On the relative discriminant of a certain Kummer field."

(19) Professor J. W. BRADSHAW: "On a method of deriving infinite products from certain infinite series."

(20) Mrs. ANNA J. PELL: "Infinite systems of linear equations with unsymmetric systems of coefficients."

(21) Professor EDWARD KASNER: "Conformal invariants of curvilinear angles."

(22) Professor FLORIAN CAJORI: "Fourier's improvement of the Newton-Raphson method of approximation anticipated by Mourraille."

(23) Dr. LOUIS INGOLD: "Note on identities connecting certain integrals."

(24) President E. O. LOVETT: "Generalizations of certain theorems concerning cases of collisions in the general problem of several bodies."

(25) Professor JOHN EIESLAND: "On minimal lines and surfaces in four-dimensional space."

(26) Professor JOHN EIESLAND: "Lie's line-sphere geometry from the standpoint of four-dimensional space."

(27) Professor E. D. ROE, JR.: "A generalized definition of limit."

(28) Professor E. D. ROE, JR.: "A new invariant function."

(29) Dr. H. H. MITCHELL: "The subgroups of the hyper-orthogonal group $HO(3, p^{2k})$."

(30) Dr. H. BECK: "Ein Seitenstück zur Moebius'schen Geometrie der Kreisverwandtschaften."

Dr. Beck's paper was communicated to the Society through Professor White. In the absence of the authors the papers of Professors Dickson, Field, Slocum, Miller, Westlund, Mrs. Pell, Professors Kasner, Cajori, Dr. Ingold, President Lovett, Professor Roe, and Dr. Beck were read by title.

Professor Dickson's first paper appeared in full in the October BULLETIN. Dr. Beck's paper will appear in the October number of the *Transactions*. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the list above.

2. The second paper by Professor Dickson determines by two methods all the groups of binary transformations of determinant unity in the general Galois field. For the more interesting types of these groups there is constructed a fundamental system of invariants. The paper has been offered for publication in the *American Journal of Mathematics*.

3. The coefficients of a binary m -ic are elementary sym-

metric functions of m groups of two homogeneous variables; and those of a p -ary m -ic will be similar functions of m groups of p homogeneous variables when the latter form is the product of m linear forms. This analogy between the binary form and the degenerate p -ary form is of fundamental importance in the study of the structure of non-degenerate p -ary forms. In the first part of Professor Glenn's paper general methods of resolving degenerate forms into linear and quadratic factors are developed, including the cases where the factors are multiple. It is found that this problem always reduces to the direct problem of solving algebraical equations in one unknown, and the necessary equations are developed. This process of resolving a degenerate form is then applied to a non-degenerate form, with the result that the latter is reduced to a certain canonical type. The following theorem is established for ternary forms: Let $f_{3m} = a_x^m + b_x^{m-1}z + \dots$ be the general ternary form, a_x^m being the binary form of order m , and so on. Let a_x^m be factored into real factors, and assume (1) that all of these factors are contained in b_x^{m-1} to a multiplicity one less than in a_x^m . Then by the processes developed we can obtain

$$f_{3m} = f_m + \mu_{m-2},$$

where f_m is the product of real linear or quadratic factors or both. Assume (2) that the satellite μ_{m-2} (of order $m-2$) is a form of the same character as f_{3m} (in that (1) is satisfied); also that its satellite is of this character, and so on. Then

$$f_{3m} = f_m + f_{m-2} + f_{m-4} + \dots + f_{2-\epsilon} \quad (\epsilon = 1 \text{ or } 2),$$

where f_{m-2k} of order $m-2k$ is degenerate into linear and quadratic factors.

4. The primary object of Mr. Carmichael's paper is to prove the existence of a fundamental system of simple analytic solutions of the system of n linear homogeneous difference equations

$$(A) \quad G_i(x+1) = x^\alpha \sum_{j=1}^n a_{ij}(x) G_j(x) \quad (i = 1, \dots, n),$$

where α is a constant and the $a_{ij}(x)$ are single-valued and

$$a_{ij}(x) = a_{ij} + a'_{ij}x^{-1} + a''_{ij}x^{-2} + \dots \quad (i, j = 1, \dots, n; |x| > R),$$

the constants a_y being such that the roots ρ_i of the equation

$$|a_y - \delta_y \rho| = 0, \quad \delta_y = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j, \end{cases}$$

are in absolute value different from each other and from zero. The proof is obtained by means of a form of the method of successive approximation and a process of induction.

If we suppose that (A) is solved for $G_i(x)$ in terms of $G_i(x+1)$, ($i = 1, \dots, n$), and that x is then replaced by $x-1$, we may write

$$G_i(x-1) = \sum_{j=1}^n A_{ij}(x-1) G_j(x), \quad (i = 1, \dots, n).$$

One result of the paper may now be stated as follows: (A) has a fundamental system of solutions

$$G_1(x), G_2(x), \dots, G_n(x) \quad (j = 1, \dots, n),$$

each function of which is analytic throughout the finite plane except at sets of points $[\beta - r]$, ($r = 0, 1, 2, \dots$), where $[\beta]$ is the set of all singularities of the coefficients $A_{ij}(x)$. If x approaches infinity along any line parallel to the positive real axis,

$$\lim G_j(x) \{ \Gamma(x) \}^{-\alpha} \rho_j^{-x} x^{-m_j} = \alpha_j,$$

where m_j and α_j are constants depending on the constants a_y and easily determined.

An existence theorem for a single equation of the first order is already known and some work (quite incomplete and unsatisfactory) has been done for a single equation of the second order; but, except for some recent important partial results obtained by means of the Laplace transformation, no further substantial progress in obtaining analytic solutions of difference equations seems to have been made.

5. In the consideration of the projective geometry upon a quadric surface

$$Q \equiv \sum_{i, \kappa} a_{i, \kappa} x_i x_\kappa = 0, \quad a_{i, \kappa} = a_{\kappa, i}, \quad |a_{i, \kappa}| \neq 0, \quad (i, \kappa = 1, 2, 3, 4),$$

with special reference to the general correlation of space

$$\Omega = \sum_{i, \kappa} b_{i, \kappa} x_i y_\kappa = 0$$

we are confronted with the question of determining a transformation

$$H: \quad \rho x_i = \sum_{\kappa} h_{i\kappa} x'_{\kappa}, \quad |h_{i\kappa}| \neq 0,$$

which when carried out simultaneously on the x 's and y 's will reduce Q and Ω to more simple forms. Algebraically this is asking for the matrix H such that, when we perform the operations

$$H'QH = \bar{Q} \quad H'\Omega H = \bar{\Omega},$$

where Q and Ω are the matrices of Q and Ω respectively, the matrices $\bar{Q}$ and $\bar{\Omega}$ will be in a more simplified form.

Dr. Burgess determines a method of reduction applicable to matrices of order n and gets the results for the cases $n = 2, 3, 4$, and 5 .

6. Professor Klein has pointed out that the form problem of the Valentiner group G_{360} in the plane would have the same import for the solution of the sextic that the icosahedral binary G_{60} has had for the solution of the quintic. Professor Gordan has shown that only an "accessory" cubic equation is necessary for the purpose. He obtains one such equation as the characteristic equation of a connex $(1, 1)$ whose coefficients are of degree six in the roots of the sextic. It is the object of Professor Coble's paper to show that a connex $(1, 1)$ exists which is of similar use and whose coefficients are of degree four only in the roots. It is shown further that there is no accessory equation of lower degree in the roots. A resumé is then made of the necessary calculations, in order to estimate the difficulty of setting out explicit formulas in the algebraic part of the problem of the sextic.

7. It was shown by Bertini* that every involutorial birational transformation of the plane into itself can be reduced to one of four types, by transforming through appropriate quadratic inversions. Of these, three were already known, but the fourth, in which the lines of the plane go into curves of order 17 with eight basis points of multiplicity six, was new. In Professor Snyder's paper the equations of these transformations are given, and some properties determined. The problem is

* "Ricerche sulle trasformazioni univoche involutorie nel piano," *Annali di Mat.*, ser. 2, vol. 8 (1877), pp. 244-286.

closely associated with that of the determination of the ninth and tenth double points of a sextic having eight given double points.

8. In this paper Professor Dickson determines a fundamental system of two invariants of the group of all binary linear transformations in a finite field. By means of a three-term recursion formula the product π_m of the irreducible forms of degree m is expressed in terms of the fundamental invariants π_1 and π_2 . Two forms are said to belong to the same class if one of them can be transformed into a constant multiple of the other by a linear transformation of determinant unity in the field. It is shown that there are as many classes of irreducible binary forms of degree m as there are irreducible factors of π_m when expressed as a function of a certain pair of invariants, the choice of which depends upon the evenness or oddness of the modulus. The cases $m < 8$ are examined completely. The paper has been offered for publication in the *Transactions*.

9. In his first paper Professor Ranum proves that in all the solutions of a system of consistent linear equations the values of a particular set of unknowns will be connected by a linear relation if and only if the omission of the coefficients of these unknowns from the matrix of the system reduces the rank of the matrix. This result is a generalization of a recent theorem of Miller's, and leads to a classification of consistent systems of rank r in n unknowns with respect to the reduction in rank caused by the omission of the coefficients of one or more unknowns. The geometric significance of this classification is explained.

10. Of the eight spheres inscribed in a tetraedron formed by four neighboring tangent planes to a developable D , seven are infinitesimal; the limiting position of the eighth is defined as an osculating sphere S of D . S is dual to the well-known osculating sphere of a twisted curve. By means of this duality, which is even more complete in riemannian than in euclidean space, Professor Ranum derives some of the properties of S , among which are the following: S is the limit of a sphere inscribed in two neighboring osculating cones of D . If D' is the edge of regression of D , and R is the rectifying curve of D' , then R is the locus of the center of S . If s is the arc of D' and if r and

ρ are its radii of curvature and torsion, respectively, then the radius r' of S is given by the formula

$$\frac{1}{r'} = \frac{d}{ds} \left(\frac{r}{\rho} \right).$$

At the point of contact of D and S their curve of intersection has in general a double point with distinct tangents. A necessary and sufficient condition that S be a fixed sphere is that D' be a geodesic line on a cone.

11. A number of papers have been written (see Glenn, "The theory of degenerate plane curves," *American Journal of Mathematics*, January, 1910, where additional references are given) on the conditions that an equation must satisfy in order that the corresponding curve shall degenerate. Professor Field's paper considers the same problem for rational curves only. The method of attack consists in expressing the coordinates of a point on the curve rationally in terms of a parameter λ , so that the curve is defined by the equations

$$\theta x = f_1(\lambda), \quad \theta y = f_2(\lambda), \quad \theta z = f_3(\lambda).$$

The coefficients are then varied so that $f_1(\lambda)$, $f_2(\lambda)$, $f_3(\lambda)$ have one or more factors which become equal. While the problem in this form is restricted, the results are correspondingly simple.

12. Dr. Rowe's paper, in four sections, consists of a treatment of the rational quartic curve in the plane. In Section I Salmon's work on the combinants of two binary quartics with some extension is used to find the equations of the covariant curves, especially those of the covariant conics, of the rational quartic. Among other results we obtain an independent analytic proof that the six flexes of the rational quartic are on a conic. Section II is devoted to finding a complete system of invariants for the rational quartic as combinant of the fundamental involution; also, by a mere permutation or cyclic substitution of letters it is possible to derive important invariants of the rational quartic from the covariant curves of Section I. In the third section a new system of invariants is obtained by considering this curve as the section of the Steiner quartic surface by a plane, the invariants being expressible in terms of the symmetric functions

of the coefficients of the cutting plane. In Section IV the connection between the two systems of invariants is established, and a method for passing from our invariants to those obtained in another way by Professor Coble in an article in the *American Journal*, volume 31, number 2, is given.

13. A linear difference system may be written in either of two equivalent ways,

$$g_i(x) = \sum_{j=1}^n a_{ij}(x)g_j(x) \quad \text{or} \quad g_i(x-1) = \sum_{j=1}^n b_{ij}(x)g_j(x),$$

where $i = 1, 2, \dots, n$; and to each of these corresponds a complete set of formal solutions. By the use of these, Professor Birkhoff proves the existence of two complete sets of solutions, which are termed principal solutions in virtue of their remarkable properties. The only singularities of the first (second) set of solutions are poles which fall to the right (left) of and congruent to the poles of $a_{ij}(x)$ [$b_{ij}(x)$], and the asymptotic form at $x = \infty$ is completely specified except along the line of positive (negative) reals. The functions $a_{ij}(x)$ and $b_{ij}(x)$ are rational functions of x .

The relations between the two sets of principal solutions is found to be expressible in terms of certain rational functions of $e^{2\pi i x}$. The asymptotic properties and these relations entirely characterize the two sets, as sets of principal solutions of a linear difference system with rational coefficients, and the number of characteristic constants is equal to the number of arbitrary constants in the given system. This leads to the formulation of the fundamental problem of linear difference equations. Finally it is shown that the solution may be made to depend on that of a like system in which the $a_{ij}(x)$ are linear in x ; this system which generalizes the ordinary gamma function is treated in some detail.

14. In this paper Professor Dickson shows that the group G_m of all linear homogeneous transformations T on m variables with coefficients in the Galois field of order p^n has the fundamental system of invariants $L_m, Q_{m1}, \dots, Q_{m, m-1}$, where

$$L_m = [m-1, \dots, 0], \quad Q_{ms} = [m, \dots, s+1, s-1, \dots, 0] / L_m, \\ [e_1, \dots, e_m] = |x_i^{p^{n_j}}| \quad (i, j = 1, \dots, m).$$

The latter determinant is multiplied by $|T|$ by a transformation T . Hence the Q_{ms} are absolute invariants. The m fundamental invariants are shown to be independent. In the proof that they form a fundamental system, use is made of the relations

$$Q_{m1} = Q_{m-11}(L_m/L_{m-1})^r + L_{m-1}^{p^n},$$

$$Q_{ms} = Q_{m-1s}(L_m/L_{m-1})^r + Q_{m-1s-1}^{p^n} \quad (s > 1),$$

in which Q_{m-1s} is to be suppressed if $s = m - 1$, while $r = p^n - 1$. Expanding L_m according to its last column, we may express L_m/L_{m-1} in terms of L_{m-1} , Q_{m-1s} , x_m . These relations enable us to express the Q_{ms} as integral functions of the x_i . The proof that the Q_{ms} are integral functions follows easily from their definition.

The solution of the "form problem" is discussed. Further applications are of the character indicated in paper No. 8. This paper has been offered for publication in the *Transactions*.

15. Professor Slocum's paper has for its main object the derivation of a general formula for the shearing deflection of a beam of arbitrary cross section, either variable or constant. The method used is based on Castigliano's application of the principle of least work. The result is embodied in the formula

$$D = \int \frac{Q}{QG} dx \int q^2 dA,$$

where

D = shearing deflection at any fixed point P ,

Q = total shear on any section due to given loading,

Q' = total shear on any section due to concentrated unit load at P ,

q = unit shearing stress due to shear Q ,

G = modulus of shear,

dA = element of area of cross section,

dx = element of length of beam.

For a constant rectangular cross section it is shown that the formula reduces to

$$D = \frac{6}{5} \int_0^l \frac{Q Q'}{AG} dx, \dots$$

and for a constant circular cross section it becomes

$$D = \frac{10}{9} \int_0^l \frac{Q Q'}{A G} dx.$$

The formula is then applied to the case where the moment of inertia varies linearly along the axis of the beam, as is approximately true of the ordinary commercial planer rail. Definite numerical applications of the formula are given in each instance, and the percentage of shearing deflection for a given ratio of length to depth is also discussed.

It is further shown that the method here used for determining the shearing deflection also leads to the formula deduced otherwise by Professor Fraenkel for flexural deflection, namely,

$$D = \int \frac{Mm}{EI} dx,$$

where

D = flexural deflection at any given point P ,

M = moment at any section due to given loading,

m = moment at any section due to concentrated unit load at P ,

E = Young's modulus,

I = static moment of inertia of cross section,

dx = element of length of beam.

It is then shown that all the ordinary formulas for bending deflection, as deduced independently from the equation of the elastic curve in each case, are simply special cases of the above. The formula is then further applied to the case where the moment of inertia varies linearly along the axis of the beam, a case not heretofore treated. Numerical application of the result is then made to the type of planer rail mentioned above.

16. The main theorem proved by Professor Miller is as follows: The necessary and sufficient condition that a given unknown has the same value in all the possible solutions of a system of consistent linear equations is that the rank of the matrix of the system is diminished by omitting the coefficients of this unknown from this matrix. Hence each unknown in a system of consistent equations must either have the same value in every possible solution or it has an infinite number of different values in the totality of possible solutions. This evidently includes the theorem that a system of linear equations has zero, one, or an infinite number of solutions. The didactic impor-

tance of the theorem was emphasized and it was pointed out that this theorem should have a place in every advanced course in algebra, as it rounds out the students' conception of a system of linear equations. It was also observed that the necessary and sufficient condition that a given unknown in a consistent system of linear equations may have the value zero in a solution of the system is that the rank of the augmented matrix of the system is reduced by the omission of the coefficients of this unknown whenever the rank of the matrix is reduced by this omission. Hence it results that the necessary and sufficient condition that an unknown may have a value different from zero in a consistent system is that the rank of the augmented matrix is not reduced by omitting the coefficients of this unknown.

17. If H represents any subgroup of index ρ under a group G , all the operators of G may be arranged so as to give ρ distinct sets as regards both right hand and left hand multiplication in the following forms :

$$\begin{aligned} G &= H + HS_2 + HS_3 + \cdots + HS_\rho, \\ &= H + S_2H + S_3H + \cdots + S_\rho H. \end{aligned}$$

The sets HS_α , $S_\alpha H$ ($\alpha = 2, 3, \dots, \rho$) are called respectively right co-sets and left co-sets of G as regards H by Professor Miller, and he observed that every right co-set is composed of the inverses of all the operators of some left co-set, and vice versa. The necessary and sufficient condition that H is invariant under a given right co-set is that this co-set is identical with some left co-set; and the necessary and sufficient condition that H gives rise to a multiply transitive representation of G , or of one of its quotient groups whose order exceeds two, is that every right co-set as regards H has at least one operator in common with every left co-set as regards H .

If G involves kn , h operators which transform H into a group having a subgroup exactly of index n_1 in common with H , then K_1 must have exactly k transitive constituents of degree n_1 and vice versa, where K_1 is composed of all the substitutions which omit one letter in the transitive substitution group K to which H gives rise. If $S_\alpha HS_\alpha^{-1}$ has a subgroup of index n_1 in common with H and if the largest invariant subgroup of H contained in this common subgroup is of index m_1 under H ,

then the corresponding transitive constituent in K_1 is of order m , and of degree n_1 . These theorems exhibit more completely the principle of duality as regards substitution groups and abstract groups.

18. In a paper presented by Professor Westlund at the February meeting the discriminant or fundamental number of the algebraic number field $k(\sqrt[m]{m})$ was determined. The object of the present paper is to show how the results thus obtained may be used for determining the relative discriminant of the Kummer field $k(\sqrt[m]{m}, \rho)$, where ρ is a primitive p th root of unity, with respect to the field $k(\sqrt[m]{m})$.

19. Professor Bradshaw illustrates a method of deriving infinite products from certain infinite series by application to Gregory's series for $\frac{1}{4}\pi$. By adding to this series a series which itself vanishes, a succession of better and better convergent series is obtained. The first term in this succession of series approaches the form of the infinite product due to Wallis.

20. Properties analogous to those for biorthogonal systems of functions are obtained for biorthogonal systems of limited linear forms in infinitely many variables. If the biorthogonal system

$$U_i(x) = \sum_{j=1}^{\infty} u_{ij}x_j, \quad V_i(x) = \sum_{j=1}^{\infty} v_{ij}x_j$$

is such that

$$L(\cdot)V_i(\cdot) = \sum_{j=1}^{\infty} l_{ij}v_j$$

is convergent for a limited linear form $L(x)$, then

$$V_i(x) = TU_i(x),$$

where T transforms every limited linear form into a unique limited linear form, and

$$L(\cdot)TL(\cdot) \geq 0, \quad L_1(\cdot)TL_2(\cdot) = L_2(\cdot)TL_1(\cdot).$$

From these properties of T Mrs. Pell shows that it has the form

$$TL(x) = K(x)L(\cdot),$$

where $K(xy)$ is a symmetric definite or semi-definite limited bilinear form. The converse theorem is also true.

If the unsymmetric limited bilinear form $L(xy)$ is such that there exists a symmetric definite limited bilinear form $K(y)$ such that

$$M(xy) = L(x \cdot) K(\cdot y)$$

is a symmetric bilinear form, then there exists a spectrum for the system of linear equations

$$U_i = \lambda \sum_{j=1}^{\infty} k_{ij} u_j, \quad V_i = \lambda \sum_{j=1}^{\infty} k_{ij} v_j.$$

In particular, if $L(xy)$ is continuous, there exists at least one characteristic number. This condition is also necessary.

21. The angles considered by Professor Kasner are bounded by arcs of analytic curves, and the conformal transformations are required to be regular at the vertices. The paper presents the conditions for the equivalence of such angles. A necessary condition is, of course, that the angles have the same magnitude θ . When θ/π is irrational this is also a sufficient condition. If, however, θ/π is rational, an infinite number of invariants of higher order exist. The case where θ is zero (horn angles) is discussed separately.

22. Fourier showed in publications of 1818 and 1831 that, if one and only one root lies between the real numbers a and b , then the Newton-Raphson method of approximation to the roots of a numerical equation $f(x) = 0$ yields with certainty successive approximations to this root if the first two derivatives of $f(x)$ do not vanish for values of x lying in the interval between a and b , and if, of those two limits, that one is chosen as the initial approximation which, when substituted for x , gives to $f(x)$ and $f''(x)$ the same sign. Professor Cajori showed in his paper that Fourier was anticipated 50 years in this by J. Raym. Mourraille, who published in 1768 at Marseilles a work entitled *Traité de la résolution des équations en général*. Mourraille arrived at somewhat simpler results by omitting Fourier's unnecessary condition that $f'(x)$ shall not vanish. Fourier was also anticipated in part of his "Rule B."

23. Because of the general nature of the symbols and sym-

bolic products of the symbolic invariant theory it is possible to apply the formulas of this theory in various special fields. In the present note Dr. Ingold interprets symbolic functions as ordinary functions of a real variable x in a definite interval, say $0 \leq x \leq 1$, and symbolic products as integrals, in the interval $0 \dots 1$, of ordinary products. With this interpretation all of the identities of the symbolic invariant theory become identities involving ordinary functions and integrals.

24. In the first section of President Lovett's paper, the fundamental formula of Lagrange's celebrated essay on the problem of three bodies is extended to the motion of a system under forces varying as any arbitrary function of the distance. This generalization is then specialized to the case of forces varying inversely as any positive power of the distance; for the latter problem the generalizations are constructed of the differential equations employed by Sundman* in characterizing the conditions for collision under the law of nature. The second section is occupied with the determination of the different cases of singularities in the motion of a system of three bodies, the forces varying as the masses and inversely as an arbitrary positive power of the distance. In the third section, the results of the first section are extended to the most general problem of n bodies, by utilizing to this end Block's† extension of Sundman's formulas to the newtonian case of any number of bodies. Finally the fourth section studies certain of the conditions for simultaneous collision in the most general case of the n body problem.

25. Professor Eiesland's first paper treats of minimal lines, congruences of minimal lines, and curved surfaces which on account of certain properties have been defined as minimal surfaces in space of four dimensions. If such a surface lies in a space of three dimensions, it becomes an ordinary minimal surface. A minimal surface of four-dimensional space is the middle surface belonging to the focal surface of a minimal (or isotropic) congruence of that space; it is generated by the ∞^2 minimal tangent spaces

$$(s+t)x_1 + i(t-s)x_2 + (st-1)x_3 + i(st+1)x_4 + F'(st) = 0.$$

* K. F. Sundman, "Recherches sur le problème des trois corps," *Acta. Soc. Sc. Fén.*, vol. 34 (1907).

† H. Block, "Sur une classe de singularités dans le problème de n corps," *Mémoires f. Lunds Astr. Obs.*, ser. 2, no. 6 (1909).

26. Professor Eiesland's second paper deals with the projection of minimal configurations into ordinary space by means of the orthogonal projection

$$X_1 = x_1, \quad X_2 = x_2, \quad X_3 = x_3, \quad X_4 = iR$$

which transforms the vertex of a minimal cone in four-dimensional space into the center of the sphere of radius R in ordinary space, while the minimal cone becomes the sphere, i. e., the cone

$$(x_1 - X_1)^2 + (x_2 - X_2)^2 + (x_3 - X_3)^2 + (x_4 - X_4)^2 = 0$$

becomes when projected into the space (x_1, x_2, x_3) (putting $x_4 = 0$):

$$(x_1 - X)^2 + (x_2 - X)^2 + (x_3 - X)^2 = R^2.$$

It is shown that Lie's line-sphere geometry follows as an immediate corollary from this theory of projection.

27. The object of Professor Roe's first paper is to explain a logical point of view, without affecting any of the facts involved, from which a greater generalization of the definition of limit can be obtained than is the case with the ordinary definition. To secure this it is necessary to distinguish between infinity and an infinite variable, and to discuss the meaning of "number" for the case in hand. The definition of limit is then so phrased that not only is the usual definition logically included as a special case, but infinity is also included as a limit, as well as any complex numbers, and even the extension of the definition to higher complex numbers may be easily made.

28. The new invariant function discussed in Professor Roe's second paper is

$$\begin{vmatrix} A_1 & A_0 & 0 & \dots & 0 \\ 2A_2 & A_1 & A_0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & A_0 \\ nA_n & A_{n-1} & \dots & A_1 \end{vmatrix},$$

where $n > 1$, $A_k = a_0 b_k + a_1 b_{k-1} + \dots + a_k b_0$, and $b_k = \lambda^k / k!$.

The function is proved to be invariant, whatever the value of λ .

It is used in the paper in finding general determinant values for the sums of the series

$$\frac{1}{1^{2n}} + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \cdots, \quad \frac{1}{1^{2n}} + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \cdots.$$

29. A hyperorthogonal group $HO(3, p^{2k})$ which is representable as a collineation group in three variables, the coefficients being marks of the Galois field $GF(p^{2k})$, has an invariant of the form

$$x_1^{p^k+1} + x_2^{p^k+1} + x_3^{p^k+1} = 0.$$

It is of order $(p^{2k} - p^k + 1)(p^k + 1)^2 p^{3k}(p^k - 1)/d$, where $d = 3$ or 1, according as $p^k + 1$ is or is not divisible by 3.

Dr. Mitchell determines the subgroups of this group for p an odd prime and k a positive integer. The least number of letters on which the group is representable as a permutation group is $p^{3k} + 1$, except for $p^k = 5$, in which case the least number is 50.

F. N. COLE,
Secretary.

THE PREPARATION OF COLLEGE AND UNIVERSITY INSTRUCTORS IN MATHEMATICS.

PROVISIONAL REPORT OF THE AMERICAN SUBCOMMITTEE OF THE INTERNATIONAL COMMISSION ON THE TEACHING OF MATHEMATICS.*

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PREFACE.

In the preparation of the following report the committee has found little or no literature dealing specifically with the training of mathematical instructors for the higher institutions of learning in the United States. The subject is, indeed, one to which little explicit attention has been paid. The committee has therefore been obliged to beat out its own path. As, moreover, the usual preparation for the teaching of mathematics is inseparably connected with the history and organization of mathematical instruction in our country, it has been found desirable to treat the subject freely and discursively.

I. SURVEY OF PAST CONDITIONS.

To obtain first an adequate comprehension of the conditions prevailing today in the teaching of mathematics, it is necessary to review briefly the past. The mathematical development in the United States during the last quarter of a century has been so unexpected and sudden, and so remarkable in its proportions, that it can scarcely be termed an evolution but merits rather to be called a revolution. Thirty years ago the contrast between the mathematical and the astronomical situation in this country was most striking. In astronomy we numbered some of the world-leaders, but in mathematics there was, perhaps, not an American known abroad for investigational work. In 1876 the Johns Hopkins University was founded, and soon afterwards the English mathematician Sylvester came to this country and started a small group of investigators. Some time prior to this Benjamin Peirce had stood forth at Harvard as a unique figure, a solitary investigator in the field of mathematics. Possibly owing to his influence, there emanated from Harvard University about 1880 the policy of sending promising mathematical students abroad that they might reap the fruits of German scholarship. To a limited extent the example of Harvard was followed by other institutions. As our mathematical students returned from abroad, stamped like many another excellent product as "made in Germany," they brought with them a knowledge of the modern advance in mathematics and an appreciation of investigation. Then it was that the mathematicians of the country were awakened so as to perceive the infinitesimal development of their science in America and the boundless advance abroad. In this manner, and largely through the leadership of Johns Hopkins

and Harvard Universities, there arose a small circle of investigators, fired with a determination to bring American conditions into some sort of correspondence with conditions abroad.*

Roughly speaking, the years 1880-90 may be said to mark a new departure in the mathematical education of the United States. Prior to this time there had been a fixity of aim and a definiteness of character in the methods of instruction. The prevailing mode of instruction was the recitation system, interspersed with a few lectures. For study a text book was used in which definite lessons were doled out daily, and upon these lessons the student was compelled to study and recite. Naturally he presented his work in the best form of which he was capable so as to make the best impression possible on his teacher. The instructor in turn devoted his time to correcting the mistakes and deficiencies of the student. Thus by study of a definite text and correction by the teacher great stress was laid upon *form*, while little or no emphasis was put upon imagination and upon the student's initiative or power of selection. It was the teacher's province to select the food; it was the student's to digest the daily allowance.

The amount of mathematics taught at this time was extremely limited. Calculus was a junior or a senior elective, and its theory was taught from the purely formal side. Spherical and practical astronomy, Salmon's Conic Sections, and at most one or two other subjects (particularly quaternions) completed the college mathematical course.

For teaching, little knowledge of mathematics was demanded beyond the subject to be taught. After graduation the ablest students were usually called upon at once to teach without further study, although after the establishment of Johns Hopkins University a steadily increasing tendency was noticeable to demand first a year or two of graduate study. But the bright student still considered it preferable to try at once his hand at teaching, looking upon this as an experience which would fit him to profit by more advanced study later. If, on the other hand, he should fail as a teacher, no time would be lost in useless study before changing his vocation.

No special preparation for teaching was then required, nor indeed was any needed; for as a student the teacher had gone

* For further information concerning the influences which made for progress, see Professor Fiske's presidential address before the American Mathematical Society, BULLETIN, vol. 11 (1904), p. 238.

already through the machinery of the recitation mill, and this produced a certain perfection and uniformity of results. The polishing of the rough and projecting edges of the raw product through the friction of daily recitation, the demand for quick and ready expression by the student when on his feet, the insistence upon clearness, accuracy, and good English in the production of work (qualities which, it should be noticed parenthetically, were necessary to the professions of the law, ministry, teaching, into which the great majority of the students of that time passed) — all this constituted in itself an admirable preparation for teaching. Whatever may have been their deficiencies in exactness, extent, and depth of knowledge, the college mathematical professors of the time were commonly adepts in teaching the text book material, and they knew how to present it so as to make it clear to the average man's comprehension. Like produces like. In consequence the young teacher, put in without special training to "sink or swim, live or die, survive or perish" according to his ability to control and guide the boisterous elements of his classes, generally fulfilled his mission satisfactorily.

The situation can be summed up by saying that the machinery was well adapted to its end and turned out its material after the model set. On the other hand, various other qualifications of supreme importance for the teacher were too often ignored or sorely neglected; for example, initiative in study, geometric and analytic imagination, rigor, power of generalization, etc. It is, therefore, not surprising that the mathematical training of that time failed completely to produce the true scholar or the great mathematician.

II. PRESENT DAY CONDITIONS.

The conditions today with respect to teaching are in part a survival of those of the past, and in part the result of the ingrafting of new ideals and methods. The earlier methods, in so far as they were not indigenous, were an inheritance from the English tutorial system. The changes have been due to contact with the mathematicians of the continent of Europe, and in particular with German mathematicians. As our students returned from study abroad and themselves became teachers, there sprang up a demand for up-to-date knowledge, for *thorough* scholarship and investigation. As a test and a certificate of some ability to meet these demands, the Ph.D.

degree speedily became a very common prerequisite for desirable college appointments. So rapid has been the change in recent years and so great the growth of the more prominent institutions of the country (especially the state institutions), that the supply of teachers and investigators has become inadequate and falls ever increasingly short of the demand.

While this condition exists in the more important institutions, the average small college west of the Alleghanies has been little touched by the awakening. Institutions of respectable standing with 800 or 1000 students also exist which are substantially in the condition of the college of twenty-five years ago, and this too, without reaching the degree of efficiency and excellence attained in the old time eastern college. Between the two extremes lie a multitude of colleges in which the new influences are felt in varying degree.

In the better institutions the mathematical progress has been attended by a remodelling of the curriculum. Calculus has become today primarily a sophomore study.* In the last two years of the college course a considerable variety of electives has been introduced, extending as far as elementary courses in the theory of functions of the real and the complex variable. Graduate courses cover a range of topics fully as wide as, and perhaps even wider than in Germany.

Speaking roughly, it may be said that up to the calculus the mode of teaching remains the same as of old. Here freer and more varied methods begin to prevail. While in most institutions it is still found advantageous to teach calculus by the recitation method, yet a large part of the class hour will frequently be spent by the teacher in discussion and explanation. The general view seems to be that about this time the student reaches a greater maturity of power, and the instruction he should receive requires therefore greater inspirational power and breadth of view on the part of the teacher. From this point on, the recitation recedes into the background, being superseded by the lecture or lecture-and-quiz method. The subject matter thereafter is no longer closely prescribed by precedent, and the course of instruction is laid out largely in accordance with the taste, training, and ideas of the teacher.

Not only is the calculus a boundary line between two styles

* At Harvard it was so recognized as early as 1870. This is indicative of certain tendencies toward the elevation of mathematical instruction in America before the advent of foreign influences.

of teaching, but it also marks a sharp change in the problem confronting the teacher of mathematics, a change almost as great and important as that from the gymnasium to the university in the German educational system. The alteration in the educational problem at the end of the sophomore year is so well known and recognized that it has been proposed by some to relegate eventually the work of the first two years of the college course to the high school; by others to separate the work of these two years into a junior college leading to an appropriate degree.

Without discussing the advantages and disadvantages of such a change we shall proceed to consider further the nature of the mathematical problem in our large colleges and universities before and after the calculus. It should be borne in mind that nothing can be said that will fit *all* of our universities and colleges, for their number is legion. Yet in the main the remarks below portray the situation correctly, for even in the academic department of Harvard University, where the lecture system is introduced from the beginning, it is accompanied by daily tests or problems in the elementary work.

In most of the leading institutions the mathematics of the freshman year has been made either elective or semi-elective in the college of liberal arts. Such, however, is its importance that classes in the elementary subjects of trigonometry, algebra and analytical geometry are very large and create a peculiar problem of administration. In the engineering schools the same condition holds for the calculus also. A sample illustration may be taken from the University of Wisconsin. Here in the years 1908-1909 and 1909-1910, there were each semester in the College of Letters and Science a total of from 300 to 330 students to be given instruction in freshman mathematics. For this purpose they were divided into eleven to thirteen sections. This situation necessitates a large amount of work of a somewhat routine character. In the engineering school, in which mathematics is required, practically the same condition prevails. The adoption of the lecture system for these lower classes seems unwise because of its failure to meet the individual difficulties of the student and to give him the repetition and drill necessary at that stage of his development.

Even in these more elementary classes where the recitation remains, one notices a change in the character of the problem confronting the teacher, especially in the eastern section of the

country.* College youths no longer form a carefully culled and picked intellectual body designed solely for the professions. The increasing educational development of the country has touched all classes, even the foreign immigrant, and has increasingly brought to college young men preparing for business pursuits. The country has been so inundated by foreign immigration as to become the "melting pot" of the nations, and the high school is the crucible. The pupils in the high schools, being in vast numbers the children of men and women who have learned the English language only after arriving in this country, mix foreign idioms with the grammar of the school. The classes are often very large (40 or 50 being not uncommon in a class), so that not much attention can be given to the individual; and the best students, in particular, suffer from receiving instruction appropriate to the mass. Moreover, the examination for admission to college has disappeared almost entirely and has been replaced by certification from the school.† For all these reasons the average student enters a university more poorly prepared than formerly, and brings with him less of the cultural influence that came with the select few of the earlier days. Under these conditions the freshman instructor finds it impossible to maintain in his classes the former rigorous standards of recitation, of good form and exposition in concise, lucid English. Consequently the recitation system no longer produces as satisfactory results as formerly, though it doubtless serves its purpose better than any substitute that has been suggested.

Subsequent to the calculus the number of pupils in mathematical classes is small. There remains chiefly the student preparing to teach in high schools and colleges, with a sprinkling of others who need mathematics for physics, engineering, or its other applications. The fundamental problem in teaching is henceforth to inculcate in the prospective teacher thorough scholarship and sound standards, which were necessarily somewhat neglected in the previous mass-instruction.

* In the newer sections of the country pioneer or unsettled conditions have prevailed largely from the beginning.

† Examinations are still required of everybody at Harvard, Yale, Princeton.

III. THE PRESENT PREPARATION FOR TEACHING AND ITS DEFICIENCIES.

Owing to the changes above described the recitation system no longer affords the prospective teacher the same drill in presentation and the same feeling for form that it once did. Under the lecture system which succeeds it the emphasis is put primarily upon acquisition and upon extension of knowledge. But the mere hearing of lectures leaves the student passive and does nothing to tax actively his powers of presentation unless supplemented by other aids. (See Section IV.) Furthermore, if the teacher himself is deficient in style and is so absorbed in acquisition and investigation that he slights the matter of presentation, the student is likely instinctively to copy his example and to remain content with crude and half intelligible exposition of his ideas. Thus it has happened not infrequently that a bright student has found nothing in his university training to correct the deficiencies of early years or his natural tendencies, and has gone out to his career of teaching only to meet with an initial disastrous failure from which it was difficult and sometimes impossible to recover.

The introduction of the lecture method into our country has indeed been so rapid that an amalgamation of the old and new modes of teaching into a well proportioned, coherent and consistent educational system has been impossible. In consequence two dangers today beset the mathematician. On the one hand, there is extreme and unreasonable emphasis on the research side of activity at the expense of form and clearness and without due regard to values, any sort of investigation being prized even though of little or no intrinsic worth. This results in the elimination of any true interest in the actual teaching side of the profession. On the other hand, we see the teacher absorbed by personal interest in the student and by the pedagogical and administrative side of his work, to the neglect or exclusion of progress and true scholarship. Both deficiencies are found in our educational system. In a few institutions the first mentioned evil may be found, while the latter is conspicuous in many.

It might be thought that the difference in the educational problems before and after the calculus would result in a corresponding differentiation of teachers into two classes, the emphasis being laid in the one class upon the imparting of knowledge and in the other upon its acquisition. In France, for example, a sharp difference is made in the preparation for

teaching in the lycée and for teaching in the universities. For teaching in the lycée it is necessary to pass the *Aggrégation*. This is perhaps the most searching examination to be found in any educational system of Europe. It includes, in addition to a test of mathematical attainments, an examination in powers of exposition. For the latter, reliance upon general education is not sufficient, even in a nation temperamentally predisposed to considerations of form and elegance, but in anticipation provision is made for special preparation and instruction. No such distinction has ever been made in our country, although in choosing men for professorships there is clearly an increasing tendency to distinguish, according to the nature of the work demanded, between those who are primarily educators or administrators and those who are scholars and investigators.

It continues to be the custom to train both these classes of men as teachers by setting them to teach freshmen, traveling upwards by the rough and thorny road of experience.* Probably the drilling of large freshman classes affords the best training in teaching that can be given to the beginner. It possesses the further advantage that the younger instructors have more nearly the same interests as the lower-class students than do the older professors, and in consequence these students are likely to feel less hesitation in exposing their difficulties to the instructor than to the professor of greater reputation. Also the professor is saved thereby from routine teaching which after many years of repetition is liable to lose its freshness and to become somewhat stereotyped, monotonous, and irksome; and he is left free for other university problems demanding his riper experience. In many institutions, however, it remains the practice for the professor to retain one elementary class (a freshman section, or more often the introductory class in calculus) for the purpose of supervision and to keep in touch with the situation in the lower classes. The contact between professor and instructor when teaching different sections of the same class often affords the instructor most valuable training and help in teaching. It seems altogether probable that with the development of our mathematical education such work of supervisory instruction will be reserved

*The extent to which this is done is shown by the fact that out of 225 graduate students of mathematics in 24 universities, 61 were registered as assistants or instructors. In addition should be counted a certain proportion of 57 scholars and fellows.

increasingly for men of fine teaching capacity but without the taste and inventiveness for investigation.

The above system of freshman and sophomore instruction is, however, not without its serious and pernicious effects. Quite commonly fifteen or more hours of this work is loaded upon the instructor, in addition to which he has numberless quizzes and "blue books" to correct, no distinction being made in the assignment of such work between the humdrum man and the scholar of talent and promise. But in many institutions the work is brightened by the assignment of a small class in some advanced subject. In rare instances (Harvard, etc.) the proportion of "desirable electives" reaches a quarter or more of his total quota. Yet in general, one result of the system of lower-class instruction has unquestionably been the draining of intellectual life out of many able men by an excess of routine work which leaves them inadequate time to carry on their study and investigation.

The handling of the huge and ever increasing mass of students also necessitates a large amount of administrative work (not connected with any particular branch of instruction). For this the mathematical profession, in particular, has been looked to. By actual count there are to-day some twenty professors of mathematics who are either deans or administrative officers of similar rank in their respective institutions; among them are men of fine ability and scholarship whose services are thereby lost to mathematical research and progress. Even when not occupying such a position, the mathematical teacher is supposed to be competent for administrative work and is prone to receive an extra large assignment of it. But the present situation, bad as it is, is better than the past, and gives hope of further betterment in the future. Probably also the seriousness of the administrative problem has been responsible for another evil beside the absorption of so much of the scholastic ability of the country. It has helped to promote the pernicious view, still too largely "held by some people of whom one would expect better things, that a man can become a good college teacher by following about the same lines as the high school teacher, learning a very little more of his subject than he has to teach, and developing important accessory qualifications, of which the one dearest to college presidents is his capacity for administrative work."

IV. SUPPLEMENTS TO THE LECTURE SYSTEM.

Diverse means are employed to correct the deficiencies of the lecture and recitation systems pointed out in the last section. Among the supplements to the lecture system the setting of problems, the seminar, and the quiz should be especially mentioned.

The first of these is used relatively but little. It was introduced at Harvard about 1880. Probably the bane of the American educational system as a whole, even to the present day, is the excessively great stress laid upon the acquisition and assimilation of material. Individual initiative in treating new questions is deserving of greater recognition and stimulus. Now the aim of problem work is to encourage this. It is to be carefully distinguished from the merely illustrative example work in our lower classes and is more akin to the use of advanced problems at English universities. It prevents mere passivity on the part of the student and, if skilfully used, will stimulate the student's imagination and pass by imperceptible degrees into real original work. Its greater use and consideration in advanced mathematical instruction in our country can therefore be recommended. In exposition it is also not without very considerable value since the student, especially when stimulated to it by his professor, may find a peculiar interest and satisfaction in formulating properly his own results. But it should be accompanied by other methods to develop more actively the power of presentation and afford proper acquaintance with periodical literature and skill in its perusal.

The "quiz" is distinctly non-German in character and origin. Though native, it is somewhat akin to a species of French conference. In the quiz a review is made of the contents of a group of lectures recently given, and this is done by discussion between student and teacher, and by sharp cross-questioning on the part of the latter. Not infrequently new material is added in this manner to the lectures already given. The advantage of the quiz is not alone to the student. The teacher is enabled thereby to keep in touch with him; and from the democratic American point of view "keeping in touch" is a cardinal element of sound teaching. The indefinite spinning of lectures by the teacher, careless whether they are being followed by the student, is foreign to our conception of education. Much could, of course, be said for the lecture system on the principle "The devil take the hindmost, O!"; for, undoubtedly, by sifting the students and casting out the weak the best talent can be most

rapidly developed. But one of the characteristic American aims is the development of an intellectual democracy rather than an intellectual aristocracy. It is greatly to be regretted that the quiz is not more frequently used, and its development more carefully studied. In the hands of the skillful teacher a quiz, say once in every four or five lectures, can be employed to instruct and aid simultaneously the weaker and the stronger pupils. Great insistence can be laid upon accuracy, clearness, conciseness, upon thorough comprehension and expression of ideas; and the slipshod work, due so often to a mere hearing of lectures, can thereby be checked. While the quiz may impede rapidity of progress in a given subject, the loss is more than compensated by the hearty cooperation and understanding which it secures between teacher and student, and by the added interest in the work.

The seminar and pro-seminar (i. e., a seminar not based upon research) are German methods adopted and adapted in the American system. In the seminar the student becomes the expositor, and oral reports are given on problems connected with his investigation or on his collateral reading. In the opinion of the committee the quiz and seminar together can be made an invaluable complement to the lecture system in the training of teachers.

Another very common aid is the mathematical club. This serves as a species of clearing house or a rallying point for the department, instructors and graduate students being brought together for discussion and report. Its advantages are very similar to those of the seminar. Here, perhaps more than anywhere else, the graduate student is anxious to show at his very best and is "tried out," to see what he can do.

In certain institutions definite courses of reading are mapped out, while in other institutions the same end is sought by collateral reading assigned in connection with regular lecture courses. The bulk of this reading is in French or German. The purpose is alike to encourage wide reading, to give facility in reading foreign treatises, and to make the prospective teacher familiar with the literature of his subject and an adept in selecting material from widely different sources.

The usefulness of the M.A. and Ph.D. theses as preparation for teaching should not be overlooked. A doctor's thesis alone has proved to be a totally inadequate preparation in written exposition. For this reason some sort of a thesis should also be required for an M.A. degree, unless some substitute is afforded,

as for example in the additional "minor Ph.D. theses" at Harvard. The experience of editors of mathematical journals reveals on the part of the average young investigator, lamentable lack of understanding as to what clear, coherent, and readable mathematical English is. Too often the editor is obliged to ask some one to assist the author by friendly criticism. Usually a remarkable improvement is noticed after the completion of three or four successive articles by the author. The responsibility for critical work of this character should not lie with the editor but with the university teachers. Nowhere can a student be taught clear exposition better than in written mathematical English, and in no department, perhaps, are the difficulties of exposition greater than in advanced mathematics. Hence in training for teaching, the use, and if possible, the repeated use of written exposition seems to the committee indispensable. The requirement of a written theme each semester in graduate courses in which no examinations are given is an expedient especially to be commended.

In many universities assistantships and fellowships are designedly used, not merely to supplement the educational force but as an aid in the training of teachers. The assistant or fellow while continuing his graduate study is asked to teach a few hours per semester; thus theory and practice are developed simultaneously. In a few institutions, notably at Chicago, there is an affiliated university high school at which selected students may also teach. In certain rare instances a peculiar cooperative scheme has been tried by the professor, who employs an advanced graduate student as his assistant in large classes. The latter is present at all class exercises, observing the methods of the teacher and taking his place in his absence. He is called upon for aid in correcting papers and to give individual assistance to the student. Thus at all stages of the class instruction he is in conference with the professor regarding the conduct of the work. This would appear to be true practical pedagogy, well worthy of further consideration and trial.

V. TEACHERS' COURSES AND TEACHERS' COLLEGES.

In recent years the pedagogic side of teaching has been recognized at the large universities and not a few colleges by the introduction of the so-called "Teachers' Courses." In a few universities courses of this character have been separated and organized into a Teachers' College, notably at Columbia Uni-

versity, the University of Chicago, the University of Cincinnati, etc. In these courses hints concerning teaching are given, attention is called to the essentials of good teaching and to the common dangers and pitfalls, and the pedagogically difficult parts of secondary mathematics are discussed. To assist the man who goes into college work it is the aim at Columbia University and the University of Chicago to "give a good knowledge of the history of mathematics, of the curricula in different countries and of the danger points which confront teachers in their early days of work." Special topics are sometimes assigned, for example, an historical study of the methods tried in the teaching of calculus.

Up to the present time the teachers' colleges have been patronized almost exclusively by elementary and secondary school teachers. They stand as a protest against the entire neglect of the pedagogical side of teaching. It is the apparent belief of these colleges that a great work can yet be done for college and university teachers as well as for teachers in the lower grades. Not only is it said sharply that "training in the art of teaching is not even a minor requirement for a degree," but it is further urged that in mathematics above all subjects it is not enough for successful teaching that a man shall know the subject he teaches; he must have power of adapting it to the average man's comprehension. For this purpose greater attention to pedagogy and to form is desirable in preparation for teaching, doubly so in a country where no fixed traditions prevail, and where the recent mathematical growth has been so rapid. Only in this way, it is asserted, can we save our schools and elementary college classes from the crude mistakes of the youthful teacher.

As yet, however, the influence of the teachers' colleges upon the preparation of college and university teachers has been felt purely as a protest. Special methods for the training of such teachers have not been there developed, and little, if anything, has been attempted in the way of actual courses for them. The whole question as to whether such courses should be given at all, and, if so, what kind of courses should be given and where — whether in such colleges or in the graduate school — remains for determination. The whole field is a virgin one and not to be developed hastily or inconsiderately. And, finally, it seems to be conceded that *dogmatic* instruction in the art and method of teaching would be injurious and disastrous in collegiate mathematics.

Little or no encroachment has, in fact, been made upon the graduate school by the professional educator. Against his view is urged the argument that there is every difference in the manner of preparing the elementary and the advanced teacher. In the elementary school the child moves in a world so different from the world of his teacher that a study of psychology and pedagogy helps to bridge the gap, but the student of eighteen to twenty is sufficiently near his teacher in maturity that such aids can be dispensed with. The deficiencies of the American college teacher are rather on the side of an accurate, clear-cut knowledge; of a failure to distinguish between the husk and the kernel and to insist without ceasing that the student too shall search for the kernel. The prevalent opinion is that definite instruction in method crushes individuality, produces regrettable uniformity, stifles life and capacity for growth. Better an ultimate success obtained after partial failure and experiment and without suppression of individuality, than a dead level success, which never rises above rule and precept. Hence in the graduate school the primary aim must ever be to build a sound mathematical foundation and to stimulate a discriminating interest and enthusiasm. The chief preparation is then to teach the student mathematics.

But after all this has been justly said, there remains with the committee the belief that the graduate instruction is arranged too exclusively with reference to research and with too little reference to the needs of the prospective teacher. A limited part of the time in the mathematical club and seminar might be devoted with profit to the more strictly professional side of the training.

Furthermore, the tendency to undue specialization in some obscure corner of mathematics — for example, of geometry or of substitution groups — should somehow be counteracted. This might perhaps be done in part by carefully planned survey courses, such for instance as Klein's masterly lectures on projective geometry or "Höhere Geometrie." In particular these survey courses could be employed to bring to the student's attention such literature and lines of study as bear directly or indirectly upon the college subjects which he is most likely to teach. And above all the need is for broad, yet incisive introductory courses in the main and vital fields of mathematics.

VI. THE AMOUNT OF GRADUATE STUDY.

Broadly speaking, the minimum requirement for teaching in college and university tends to become the equivalent of the master's degree. But in the better institutions it is recognized that a year or a year and a half of graduate training is by no means enough for a permanent college or university position, and the possession of the Ph.D. degree is therefore made a *sine qua non*. The situation thus created is in many ways unsatisfactory in regard to both degrees.

On the one hand, the demand for graduates with the doctor's degree has had increasingly the effect of making the Ph.D. degree a sort of college teacher's certificate, and has acted in a generous measure to cheapen the degree. Too many doctors have doubtless been turned out who have no actual interest in research and seek the degree solely for its commercial value. It has been the rule rather than the exception that the holder of the Ph.D. degree fails to go on to further research.

On the other hand, the situation with respect to the master's degree is even worse. It is in no way an adequate certification for college teaching. In many institutions it has been regarded as essentially a degree for secondary teachers. In other institutions it is used with one class of students as a stepping-stone to a higher degree, with another class of students who are found not to possess the inventiveness and fertility of mind necessary for research it is both a recognition and discharge. It is indeed difficult to say for just what the M.A. degree does stand. The committee submits it as a question worthy of consideration whether the M.A. degree might not be so administered or developed as to become a desirable teacher's degree or certificate in distinction from the Ph.D. degree as a certificate of research. Thus it could be used as a recognition alike for college teachers without the doctor's degree and for secondary teachers of decidedly superior capacity. Possibly in this way the Ph.D. degree could be relieved of some of the heavy load which it is now compelled to carry.

The present sharp differentiation of college mathematicians into two classes, the holders of the master's and of the doctor's degrees is in many ways a most unfortunate one. Many men have stopped with the former who are capable of proceeding much further. The committee wishes to record its emphatic belief that every man should be encouraged to study just so far as his ability and taste may qualify him, without stopping at

the line of either degree. Many a teacher without the kind of ability necessary for research can yet be encouraged to become a thorough scholar in some definite line, instead of looking to administrative office as his career.

More systematic encouragement for study after the attainment of the doctor's degree is also a desideratum. The introduction of the German *privat-docent* system into this country is recognized to be impossible and totally foreign to the American spirit of financial independence after graduation. Yet some means should certainly be devised for recognizing better the more talented few and giving them opportunities for study commensurate with their ability. As before pointed out, the life and ambition of many young instructors have unquestionably been stifled by the load of fifteen or more hours of class room work, with the accompanying written exercises for correction. In the case of the very ablest young instructors some financial provision for maintenance should be made with a reduced number of hours of work. The prevalent American practice is to relieve the instructor of his heavy hours of work only when he is about forty years of age, or after he has proved himself a pronounced success by virtue of his ability or attainment. In opposition to this it cannot be too much emphasized that the age of preparation for intellectual work of high order is from twenty to thirty, or to thirty-five at the latest.

VII. THE SUPPLY AND APPOINTMENT OF TEACHERS.

The same general considerations which affect the supply and quality of teachers in other departments apply also to the teachers of mathematics. The serious feature in the situation today is a still increasing deficiency in supply relatively to the demand. This necessitates also to a considerable degree a deficiency of quality, since places must be filled by such persons as can be obtained. The deficiency in supply is primarily a result of the enormous growth of the educational system of the country, but the deficiency in quality is due also to the rapid advancement of the mathematical profession in standards and knowledge. On the absolute scale the number of good mathematical teachers in the United States has never been greater, yet never also has the shortage been greater. The expansion of the country and its increase in population has outstripped growth in education, and the outlook is that the situation will become even more acute in the immediate future. Not only must the prospective growth

of the country, especially of the middle and far west, be taken into account, but also the awakening and development of our neighbors, China and South America. This expansion at home and abroad affords a great opportunity for the engineer. It attracts and doubtless will continue to attract a large number of the ablest mathematical students away from abstract mathematics and teaching into the practice of the engineering and allied professions.

The attractions of these professions in themselves are enhanced by the remuneration. The financial rewards for the successful engineer are incomparably greater than those for the mathematician. Twenty-five years ago the standard salary for the full mathematical professor in our best colleges was \$2500, the variation to either side of this figure being but slight. Today the corresponding figure is \$3000, the variation ranging from \$2500 in small cities to \$4000 or \$5000 for a university situated in a large metropolis. For assistant professors \$1500 to \$2500 is the current compensation. For the successful engineer the compensation must be reckoned in five place figures.

These salaries may very possibly appear adequate to the foreign professor not acquainted with American life and social conditions, but they are notoriously inadequate to the legitimate demands of life in this country. The high price of rents and cost of living, the cost of travel necessary in a country of large distances to attend conventions and to keep in touch with mathematicians at home and abroad, and many expenditures due to prevailing social conditions are recognized to press hard upon a college professor. If he has a family of three or four children, the financial pinch is likely to be felt daily and bitterly, and anxious thought must be spent upon the mode of spending one's income. Furthermore, the mathematician by the very nature of his subject is cut off from the usual modes of supplementing a professor's salary open to his colleagues, as for example by writing or by the delivery of public lectures, by legal and expert advice, by chemical or biological analysis, and so on. His only resource may be to write a text book in one or more of the four elementary subjects in which there are large college classes. Hence we see issued year after year a large number of such books, the majority of which resemble one another like peas in a pod, and contribute little or nothing to mathematical progress. Rarely does the mathematician have both time and knowledge to write—for such fame and glory

as there may be in it—the unremunerative advanced mathematical text book or treatise so much needed in the English language.

With the obvious financial consideration there acts also another powerful force pulling men away from mathematical teaching. It is felt that the engineering professions give a chance to do “something worth while.” In our rapidly developing material civilization the shallow view often prevails that the man of action stands on a higher plane than the man of ideas. The latter also is invaluable for highest development.

This side-tracking of genuine mathematical talent to engineering work is most seriously felt in applied rather than in pure mathematics. Precisely here, where the American mind might be expected to scintillate with flashes of genius, there is a real poverty of talent. While we boast our late-lamented Gibbs,—born like many a precious flower to blush unseen,—it must be confessed that his career is indicative rather of the interest and talent which might exist in applied mathematics than of that which does exist. In support of this statement it suffices to point out how few university centers there are in the United States where a first class training in applied mathematics can be obtained. The diversion of students to engineering is not solely responsible for this. It is in part a consequence of past influences when mathematics was pursued in our country as a branch of logic and a purely deductive science. A tendency to the purely formal side of mathematics can be noted from the days of Benjamin Peirce on. It is therefore not much to be wondered at (though much to be regretted) that American mathematical research has inclined so much to this formal side; for example, to multiple algebras, postulational and axiomatic foundations, theory of groups, etc. Hence, there is need of great insistence today that applied mathematics shall be more abundantly pursued and for a longer period; that men of sound mathematical training shall be brought into touch with vital physical problems.

The current method of filling vacant positions is the Socratic one of question and answer. The professor in charge of the department or some one of its divisions first makes necessary inquiries, writing particularly to the mathematical centers where the graduate students are most likely to congregate. When practicable, he supplements the information by personally meeting the candidate under most favorable consideration and thus

gains first-hand impressions. If the position to be filled is a major and not a minor one, the candidate is frequently invited to come to the university for a visit, to inspect and to be inspected. The considerations governing a choice of candidates are discussed with the president, or president and dean, and usually the choice of the professor is accepted as a result of the conference. In some cases the selection must be approved by an academic council consisting of some or all of the full professors. Under normal circumstances the subsequent ratification by the board of regents or trustees is a matter of form, since the selection of instructors is properly deemed a prerogative of the president and board of instruction.

When the head of a department is to be chosen, a common practice is to appoint a committee of from three to five members, inclusive of the president or dean and containing representatives of other departments than the one concerned. This committee then seeks for information in all possible ways.

Applications for a vacant position are often forwarded by those who would be glad to secure the appointment. To a large extent these applications are waste paper. Wider publication of vacancies has been advocated by some; but it is the almost universal experience of those who have the appointments to make that a too wide advertisement results in a bothersome and useless hoard of applications from men, the number of whose submitted testimonials varies inversely as the square of their fitness. Information privately solicited is trusted more than the public testimonial. In the case of well-known men a hint of their availability is as effective as any formal application, but in the case of minor appointments information supplied by a really worthy applicant is sometimes a most important factor in the decision for his appointment.

To fill vacancies in small or less important colleges, the teacher's agencies are often consulted, as by many normal and secondary schools. In such cases the standing of the college and the salary are not sufficient to attract the better trained men secured by the methods just described. These institutions rely particularly upon their recent and brightest graduates to fill the vacancies. To a great extent the practice of "inbreeding" by selection from the alumni of the institution has been current in our country, and this has been true even of some of the largest and most important of our institutions. When carried to excess, the practice cramps and narrows the development of

the college or university. It is increasingly felt that the best results are obtained by generous infusion of instructors having other ideas and training than those characteristic of the institution itself.

It cannot be said that mathematical proficiency always dictates the appointments. Not infrequently there are better men available. Yet this is no doubt less often the case than is realized, since frequently appointments which are severely criticised have been made only after better men have been sounded and have declined to be considered for the position. Other qualifications than mathematical proficiency must receive consideration, such as enthusiasm and the ability to stir mathematical interest in students, personality and the genius to gather and keep coherent a group of promising instructors, ability to develop a department so as to make it useful and respected in the institution, and so on. Qualities such as these will doubtless weigh more in the future than in the past, and administrative ability will be less considered owing to the increasing separation of business work. Doubtless in the past many sins of bad mathematical appointments must be laid to considerations of administrative ability.

Two qualities may be noticed as especially needed by the American teacher. The first is a broad, liberal culture. The pursuit of mathematics in itself is doubtless a narrowing one. There is perhaps no science of which the development has been carried so far, which requires greater concentration and will power, and which by the abstract height of the qualities required tends more to separate one from daily life. A wide liberal culture therefore is eminently desirable for the establishment of that attractive personality so necessary for the best success in teaching.

The second quality, moral fibre and force, is demanded largely in all institutions but especially in the small American college where teacher and student come into particularly close personal relations. No just report concerning the selection and distribution of mathematical teachers can be given without some reference to this fact. The denominational origin of most of the eastern and older colleges accounts for the emphasis laid upon the possession of these qualities. While denominational lines have broken down and tended to disappear, and accordingly religious distinctions have been forced increasingly into the background in college and university education, the moral

ideal nevertheless survives, receiving perhaps a greater consideration than in any other country except England. In the junior college—to designate thus the first two years of the college course—the student is still young and immature and in a formative period. Here the public claims that the student has as much a right to the care, time, and thought of the teacher as has investigation. It insists, even in the large state universities, that the child shall be cheated neither out of its intellectual nor out of its moral birthright. The mathematical teacher must with rare exceptions begin his career either in the small college or in the junior college of the large university. He consequently must have the problems of the junior college at heart, and not neglect his students for his own investigation. Not a few talented men fail altogether because of such neglect or lack of interest and thereby close their mathematical career.

In the graduate school the considerations of the last section fall mostly into abeyance since the student is now sufficiently mature to criticise rather than copy the deficiencies of his instructor. Intellectual leadership and vitality are here the crucial question. With increasing separation between the junior and senior colleges there comes an increasing difference in the kinds of teachers required.

The question, "From what class of society are our mathematical teachers drawn," is a complicated one, not to be answered by statistics. Have they come up through poverty and hardship, liking mathematics for its difficulties? Do they come from the great middle classes, or from well-to-do cultured homes? Undoubtedly all classes contribute, but the chief source is probably the second, especially that section of the middle classes which is more poor than rich and the section which lies at the fringe of the cultured classes. To men from these homes the teacher's profession offers improved social conditions and increased opportunity. Its financial deprivations are borne with sufficient philosophy and complaisance, for the love of intellectual pursuits is the life of the teacher.

The large number of mathematical teachers coming from the very small college, where the curriculum is cramped and where mathematics is one of its chief components, should be especially noted. Probably these colleges furnish an abnormally large percentage of the mathematical students for the graduate school.

VIII. CONCLUDING REMARKS.

In conclusion, the committee abstains from prophesy or the formulation of any definite and ideal program for the training of the advanced mathematical teacher. The situation regarding such training is still chaotic, and there is no method yet apparent of evolving order. The aim of this report has been, rather, to point out the causes, results, and deficiencies of the present method of training (or lack of method), and it will be enough if the committee shall have succeeded in arousing interest and stimulating thought upon the subject.

Certain general points may, however, be noted by way of summary or emphasis.

The introductory courses in calculus, as already indicated, forms really the beginning of the training today. Prior to this the study of mathematics is pursued chiefly as a practical and disciplinary training for the general college body, while thereafter it is pursued by the mathematician for its own sake.

The instruction of the mathematician and physicist for a longer time together in identical courses would undoubtedly have its advantages. Certainly the greater intermingling of pure and applied mathematics is desirable. The extent to which they have been separated is astonishing. The pure mathematician is prone to be either helpless or ignorant before the applications, while the applied mathematician has rarely a good grip on modern mathematical principles. Not in a divorce from related subjects, but rather from the inspiration of contact with them is the continued life and progress of that science to be sought which is the most perfect and coherent development of human thought.

The committee expresses the further wish that the early training of the mathematician on the purely mathematical side shall not be too narrow. The four great fields, analysis, mechanics, geometry, algebra (number fields, etc.) should early be represented strongly in their *main* essentials and characteristics.

The matter of training in clear, coherent, and interesting presentation both oral and written should not be overlooked, as is sometimes done by the teacher, but he should be willing to give his time generously to their inculcation. A more abundant practice therein should be afforded for the future teacher and investigator.

Probably greater cooperation between the experienced teacher and the beginner regarding the conduct of elementary college

instruction is needed. This might be gained by more abundant discussion of the problems involved, by helpful suggestion and mutual visitation of classes, and occasionally even by the joint conduct of a class.

Finally, the amount of routine instruction, at least for the most promising of our younger college and university investigators, should be so regulated as to further rather than suppress their individual development. On the other hand, the man with a gift for teaching, though not for investigation, should be encouraged to obtain a broad and comprehensive knowledge of mathematics, and, obtaining this, should not be cashiered for failure to "produce" nor be spoiled as a teacher by being trimmed into a very mediocre investigator. The difference between the two kinds of gifts — the power to teach and the ability to extend the frontiers of our science — should be more clearly recognized. It is for a free and not a standardized development that we plead; and, above all, for greater freedom and leisure for the most able of our younger instructors that they may achieve the best that is in them.

VECTOR ANALYSIS.

Vector Analysis. By JOSEPH G. COFFIN. New York, John Wiley and Sons, 1909. xvii + 248 pp.

Einführung in die Vektoranalysis mit Anwendung auf die mathematische Physik. Zweite Auflage von Prof. Dr. RICHARD GANS. Leipzig, Teubner, 1909. x + 125 pp.

Die Vektoranalysis und ihre Anwendung in der theoretischen Physik. Von Dr. W. v. IGNATOWSKY. Theil I: *Die Vektoranalysis*, 1909, vi + 112 pp. Theil II: *Anwendung der Vektoranalysis in der theoretischen Physik*, 1910, iv + 123 pp. Leipzig, Teubner.

A GLANCE at books of the above type calls attention to the unsatisfactory position that vector analysis still occupies in our courses of study. We find here discussions of the most elementary kind mixed with other discussions requiring a considerable knowledge of mathematics and physics. To the student unacquainted with the branches of science involved, vector analysis is hardly intelligible. To the student acquainted with them, it seems superfluous as leading merely to results already

known. It seems that this duplication might be avoided. In fact, vector analysis is not a separate subject but is part of many other subjects. Vectors and their addition could be given in elementary geometry, their multiplication in trigonometry, the differentiation and integration in calculus, and the physical applications in connection with the various scientific subjects to which they apply. It is through some such fusion that vector analysis will ultimately assume its proper everyday place in mathematics and cease to be an ornamental means of stating results already known.

The first of the above books will no doubt appeal strongly to students who desire a simple yet thorough treatment of vector analysis and who do not have the time or inclination to read a larger book such as that of Gibbs-Wilson. The notation is that of Gibbs except that the cross of the cross product is made smaller and placed above the line. Thus the cross product is $\mathbf{a} \times \mathbf{b}$ instead of $\mathbf{a} \times \mathbf{b}$. Vectors are printed in black type, scalars in italics. This distinction is easy enough in print but it would interest many of us to know how it is to be effected in writing on paper or blackboard.

At the beginning of each chapter the theory is developed. This is followed by applications to mechanics and physics and the chapter ends with examples for solution. The first four chapters contain the addition and multiplication of vectors and the differentiation with respect to a scalar, with application to centers of gravity, work, moments, geometrical loci, curvature, and the dynamics of a particle. The following chapter contains the differential operators and their physical interpretations with applications to conservative systems and perfect differentials. The remaining two chapters contain integration theorems, applications to electrical theory, the dynamics of a system, and hydro-mechanics. The book ends with an appendix containing a comparison of systems of notation and a summary of formulas.

The examples for solution are ample and without being puerile are simple enough to be solved by the average reader interested in vector analysis rather than the particular example. The proofs are given evidently with the idea of convincing the reader that the results are true. Definitions of technical terms are given and physical facts clearly stated so that the book might be read by students unfamiliar with the subjects discussed.

On two points the author is rather enthusiastic. He frequently repeats that "all quantities representable by vectors

obey the parallelogram law." The incautious reader might apply this to finite rotations. Again he asserts that all natural phenomena are continuous and hence it is not necessary to complicate the discussion by consideration of discontinuities. Avoiding perhaps the realm of molecular mechanics, the point may be well taken. Unfortunately however we must represent physical phenomena by mathematical expressions which are sometimes discontinuous and which give by mathematical processes correct results only in regions for which those processes are valid. Hence many theorems, stated without restriction, require in their application that some attention be given to considerations of continuity. Yet to the thinking student such remarks are perhaps unnecessary, and vector analysis is not likely to interest those altogether averse to the process of thought.

The book of Professor Gans is a second edition of a work too well known to require detailed discussion. The principal addition is a chapter on strains. These are discussed by means of the ellipsoid and no use is made of the dyadic.

In the preface to his book Ignatowsky expresses his belief in the value of presenting a scientific subject in as many ways as possible. That he has adopted a viewpoint somewhat out of the ordinary is shown by the fact that he scarcely mentions coordinates in the mathematical part of his work. And when they are finally introduced in the discussion of strains it is excused by the existence in the field of a natural set of directions. We must admit that some of the discussions are not as simple as the corresponding ones usually given in coordinates. Yet that may be a criticism of the book and not of the method. It is a question whether the apparent advantage of using in demonstrations concepts not contained in the final theorems indicates more than that the best method of demonstration has not yet been devised.

The scalar product of $\mathbf{a}$ and $\mathbf{b}$, written $\mathbf{ab}$, is defined in the usual way. The vector product, written $[\mathbf{ab}]$ is introduced by integration. A directed surface element $d\mathbf{f}$, i. e., an element of surface whose border is described in a given direction, is represented by a normal vector whose length equals the area of the element and whose direction is determined by the corkscrew rule. Two vectors $\mathbf{a}$ and $\mathbf{b}$ are placed so that they form a par-

allelogram with the arrows pointing in the same direction around it. The vector product is then defined as the integral of $d\mathbf{f}$ over the parallelogram, the sense of $d\mathbf{f}$ being indicated by the arrows on the border. By integration over surfaces of particular forms the laws of combination of vectors are deduced.

The operations d/dt and ∇ are introduced as corresponding linear and space processes. If A is a function of position, scalar or vector, and $\nabla()A$ represents any one of the operations $\nabla \cdot A$, $\nabla \times A$ or ∇A , which under the circumstances has a sense, the definition of ∇ used by Ignatowsky may be written

$$\nabla()A = \lim_{V \rightarrow 0} \frac{\int_F A()d\mathbf{f}}{V},$$

where F is the surface surrounding V . This is Maxwell's view of ∇ as an average over the surface referred to the volume as a unit. From this definition by integration over regions of special form the properties of gradient, divergence, etc., are derived.

The method of obtaining integration theorems from this definition of ∇ is very crudely shown by writing the above equation

$$\nabla()A = \frac{\int_{dF} A()d\mathbf{f}}{dv}.$$

We may write this

$$\nabla()A dv = \int_{dF} A()d\mathbf{f}$$

and by integration get

$$\int_V \nabla()A dv = \int_F A()d\mathbf{f},$$

where F is the exterior surface, since integration over the interior surfaces surrounding volumes dV occurs twice with normals in opposite directions.

The book contains an interesting chapter on multiply connected surfaces and another on lamellar and solenoidal fields. Then after the theory is practically developed, rectangular co-

ordinates are introduced and used in the discussion of dyádis and strains.

The second part, on applications to physics, begins with a very interesting discussion of the general principles of dynamics, from which it is interesting to note the absence of Lagrange's equations and Hamilton's principle. This is followed by applications to electrical theory, a short chapter on the Lorentz theory of electrons, and one on the transmission of light through crystals.

H. B. PHILLIPS.

NOTES.

THE fourth regular meeting of the Southwestern Section of the AMERICAN MATHEMATICAL SOCIETY will be held at the University of Nebraska on Saturday, November 26. Titles and abstracts of papers should be in the hands of the Secretary of the Section as early as November 10.

THE seventeenth annual meeting of the AMERICAN MATHEMATICAL SOCIETY will be held in New York December 28-29. The winter meeting of the Chicago Section will be held at Minneapolis, Minnesota, in affiliation with the sixty-third meeting of the American Association for the advancement of science, on December 29-30. Titles and abstracts of papers to be presented at these meetings should be in the hands of the respective secretaries by December 10.

AT the meeting of the London mathematical society held on June 9, the following papers were read: By W. H. YOUNG, "A new method in the theory of integration," and "On semi-integrals and oscillating successions of functions"; by G. T. BENNETT, "The composition of finite screw displacements"; by M. J. M. HILL, "Note on the theory of linear differential equations"; by P. MILNE, "The generation of cubic curves by apolar pencils of lines."

AT a meeting of the International Commission on the Teaching of Mathematics held at Brussels on August 8, 1910, thirty delegates were present. These included Professor KLEIN, of Göttingen, President; Sir GEORGE GREENHILL, of London, Vice-President; and Professor FEHR, of Geneva, Secretary.

Representatives were present from Germany, France, Belgium, Switzerland, Russia, Holland, Spain, Hungary, England, and the United States. In the absence of the American Commissioners, Professor C. B. UPTON was delegated as their representative. Reports were presented showing the status of the work in the various countries, together with the methods of procedure. Germany's plan is to issue a considerable number of monographs prepared by specially selected individuals. England has arranged to follow the same plan. The work in France is organized somewhat as in America, with a number of committees and sub-committees. Owing to the highly centralized system in France it has taken some time to secure permissions for undertaking all of the investigation. These permissions have now been granted and the plan in that country is to publish the reports at one time in about five volumes. Holland has practically finished the work for that country, and the reports will soon be published. Spain has already published two or three sub-reports, and in the various other countries a number of reports will appear in the course of a few months.

The investigation in the United States has been carried on by means of committees and sub-committees, and most of the reports have now been practically finished. It is expected that they will be edited during the present year and that the Bureau of Education at Washington will be able to arrange for their publication.

The next general meeting of the Commission will be held next year in Europe, the place and date being as yet unsettled. It is expected that the International Congress at which the reports will be offered will be held the latter part of August, 1912, at Cambridge, England.

A full report of the Brussels meeting appeared in the September number of the *Enseignement Mathématique*.

At the meeting of the British Association for the advancement of science which was held from August 31 to September 7, considerable discussion took place on the possibility of avoiding highly technical papers, and some progress was made. A notable feature of the session was the number of joint meetings of sections with the natural result that subjects of general interest were more freely discussed. In its résumé of the work of the Association the *London Times* (September 9, 1910) says that in Section A, mathematical and physical science, the presidential address by Professor E. W. HOBSON, F.R.S., on the "Tendencies of modern mathematics," proved exceedingly

interesting and it put the point of view of the pure mathematician perhaps more distinctly than it has ever been put before a general audience. Sir J. J. THOMSON's paper on positive electricity brought out one of the most important and interesting announcements of the Sheffield meeting; he stated that his experiments on positive rays led him to infer properties in the positive electron corresponding to those previously discovered by him in the negative electron. Among other papers which attracted special attention in this section were those of Sir W. RAMSAY and Dr. R. W. GRAY on the "Molecular weight of radium emanation"; by Dr. J. A. CROWTHER on the "Number of electrons in the atom"; and the very important paper by Professor HICKS on the "Relation of spectra to the periodic series of the elements." Professor Hicks' paper has proved useful in bringing together the pure mathematician and the chemist and physicist, who of late years have tended to drift apart. In any mention of papers that by Dr. CHREE on "Atmospheric electricity," which excited admiration for its excellence and clearness, must not be omitted. In this section there have been three discussions, in each of which some man of science was asked to introduce a subject. This introduction was not necessarily done by means of a paper containing new results, but was a résumé of the state of knowledge at the present day. This is considered to be one of the most important and valuable parts of the work of the section, and it will be developed in the future. There was quite a good attendance of working physicists in Section A, and their informal conversations have proved very helpful. In short, this section, which is necessarily a highly specialized one, has thoroughly justified itself this year.

THE annual meeting of the French association for the advancement of science was held at Toulouse, August 1 to 6, under the presidency of Professor GABRIEL, with Mr. E. BELOT as chairman of the section of mathematics and astronomy. Besides a number of papers in astronomy, the following mathematical papers were read: E. LEBON, "Darboux and Picard;" M. LITRE, "The problem of the composition of rotations," and "The principle of Galileo;" G. TARRY, "Note on hyperbolic angles;" M. JOLIVET, "A new demonstration of Fermat's theorem;" A. GÉRARDIN, "Notes on the theory of numbers;" F. BOULAD, "Arithmetic resolution of a certain quintic equation;" E. N. BARISIEN, "Resolution

of the equation of the third degree;" A. PELLET, "Concerning equations having real roots;" M. FONTANEAU, "D'Alembert's principle and its application to hydrodynamics." The next meeting will be held at Dijon, the chairman of the section of mathematics again being Mr. Belot.

THE annual list of American doctorates published in *Science* presents for the academic year 1909-1910 353 names, of which 178 are credited to the sciences. The following 23 successful candidates offered mathematics as major subject (the titles of the theses are appended): M. J. BABB, Pennsylvania, "The second category of the groups of order 2^m which contain self-conjugate sub-groups of order 2^{m-1} "; Miss E. R. BENNETT, Illinois, "Primitive groups with a determination of the primitive groups of degree 20"; H. B. CURTIS, Cornell, "Hyperabelian functions expressible by theta series"; F. F. DECKER, Syracuse, "On the order of a restricted system of equations"; G. C. EVANS, Harvard, "Volterra's integral equation of the second kind with discontinuous kernel"; A. B. FRIZELL, Kansas, "Foundations of arithmetic"; F. T. H'DOUBLER, Wisconsin, "On certain functional equations"; T. H. HILDEBRANDT, Chicago, "A contribution to the foundations of Frechet's calcul fonctionnel"; F. L. HITCHCOCK, Harvard, "Vector functions of a point"; J. E. HODGSON, Johns Hopkins, "Orthocentric properties of the plane directed n -line"; J. K. LAMOND, Yale, "Improper multiple integrals depending on a parameter"; H. F. MACNEISH, Chicago, "Linear polars of the k -hedron in n -space"; E. J. MILES, Chicago, "The absolute minimum of a definite integral in a special field"; H. H. MITCHELL, Princeton, "The sub-groups of the linear group $LF(3, p)$ "; U. G. MITCHELL, Princeton, "Geometry and collineation groups of the plane $PG(2, 2^2)$ "; Mrs. H. B. OWENS, Cornell, "Conjugate line congruences of the third order defined by a family of quadrics"; Mrs. A. J. PELL, Chicago, "Biorthogonal systems of functions with applications to the theory of integral equations"; R. S. POND, Kansas, "Collineations in space of four dimensions"; J. E. ROWE, Johns Hopkins, "A complete system of invariants for the plane rational quartic curve, and other facts in regard to rational curves"; H. A. RUGER, Columbia, "The place of analysis in the curve of efficiency"; E. W. SHELDON, Yale, "Critical revision of de Haan's tables of definite integrals, two volumes"; L. L. SILVERMAN, Missouri, "On various defini-

tions of the sum of a divergent series"; H. W. STAGER, California, "On numbers which contain no factors of the form $p(kp + 1)$."

The largest list for any preceding year contained 21 names (1905).

THE Danish academy of sciences announces the following prize problem, for the best solution of which the academy's gold medal will be awarded:

To explain a method of transformation of an asymptotic series into a convergence series which will be applicable to most asymptotic series already known. In particular, the question should be discussed, what interpretation should be given to the expansion when the real variable is supposed complex within a convenient domain.

Competing memoirs must be written in Latin or any modern language of northern Europe, and should be sent to the secretary, Professor H. G. Zenthen, Copenhagen, before October 31, 1911.

THE physico-mathematical society of Kasan announces that two Lobachevsky prizes of 500 roubles each will be awarded by the society in 1912. The first will be given to the author of the most meritorious work on non-euclidean geometry which has been published during the last six years in one of the following languages: Russian, French, German, English, Italian, or Latin, and submitted to the society.

For the other prize the society announces the problem: The study of the general integrals of the equations of Painlevé (differential equations of the second order and of the second degree, whose general integrals have fixed critical points). A detailed study of one of the types of these equations is desired. Literature: Painlevé, *Comptes Rendus*, volume 126, *Bulletin de la Société mathématique de France*, 1900; Gambier, *Acta Mathematica*, 1909.

Competing memoirs either printed or in manuscript, should be sent to the society before November 4, 1911.

CAMBRIDGE UNIVERSITY. (Michaelmas term (M) begins October 13, Lent term (L) January 19, Easter term (E) April 24.)—By Professor E. W. HOBSON: Integral equations, three hours (M); Fourier's series and other representations of functions by series and integrals, three hours (L); Spherical harmonics and allied functions, three hours (L).—By Professor

Sir G. H. DARWIN: Gravitational potential and attractions (M); Lunar theory (Hill's method) (L).—By Professor Sir R. S. BALL: Celestial mechanics, three hours (M); Spherical astronomy, three hours (L).—By Professor Sir J. LARMOR: Electricity and magnetism (M); General electrodynamic and optical theory (L); Thermodynamics and molecular theory (E).—By Dr. H. F. BAKER: Introduction to theory of functions, three hours (M, L); Algebraic geometry (three dimensions), three hours (M).—By Mr. J. H. JEANS: Higher dynamics (M); Theory of probability with reference to physical problems (M); Kinetic theory of gases and matter (L).—By Mr. R. A. HERMAN: Hydrodynamics (M, L, E); Differential geometry (L).—By Mr. H. W. RICHMOND: Algebraic plane geometry (M); Synthetic geometry (L).—By Mr. T. P. A. BROMWICH: Theory of potential with electrical applications (M); Electric waves (L, E).—By Mr. W. E. BARNES: Linear differential equations (L).—By Mr. R. R. WEBB: Vibrations and sound (M).—By Mr. A. BERRY: Theory of differential equations (M); Elliptic functions (L, E).—By Mr. G. T. BENNETT: Line geometry (L).—By Mr. J. H. GRAËCE: Theory of numbers (M); Theory of invariants (L).—By Mr. BIRTWISTLE: Hydromechanics (L).

PROFESSORS G. A. MAGGI and P. PIZZETTI, of the University of Pisa, have been elected national members of the Accademia dei Lincei; Professor J. H. POYNTING, of the University of Birmingham, has been elected foreign member, and Professor F. SEVERI, of the University of Padua, corresponding member of the same academy.

PROFESSOR V. VOLTERRA, of the University of Rome, has been elected foreign member of the royal society of London.

PROFESSOR U. DINI, of the University of Pisa, has been elected president of the Italian society of sciences (the "forty"); Professor T. LEVI-CIVITA, of the University of Padua, has been elected to membership in the same society.

DR. M. CARRUS, in charge of the mathematical conferences at the University of Besançon, has been appointed professor of the calculus.

DR. A. VILLAT has been appointed in charge of mathematical courses at the University of Caen, and Dr. L. AUBRY of those at the University of Algiers.

A BRONZE statue of Lord KELVIN by Mr. Bruce-Joy is being erected at Belfast, Ireland. The figure holds a gyroscope and by its side is a Kelvin compass.

MR. FRANK WATSON DYSON, F.R.S., astronomer royal for Scotland and formerly chief assistant at Greenwich, has been appointed astronomer royal in succession to Sir W. CHRISTIE, who will retire on October 1.

ON the occasion of the Sheffield meeting of the British Association, the University of Sheffield conferred the degree of doctor of science on the chairman of each section; Professor E. W. HOBSON, of Cambridge University, was chairman of section A, mathematics and physics.

PROFESSOR OSKAR BOLZA, of the University of Chicago, has been appointed honorary professor of mathematics in the University of Freiburg.

PROFESSOR E. W. STANTON, of the Iowa State College, has been elected acting president of that institution.

PROFESSOR R. J. ALEY, of Indiana University, superintendent of public instruction of the State of Indiana, has been elected president of the University of Maine.

AT the University of Chicago, Professor L. E. DICKSON has been promoted to a full professorship of mathematics, and Dr. A. C. LUNN to an assistant professorship of mathematics.

AT Purdue University Professor WILLIAM MARSHALL has been promoted to an associate professorship of mathematics, and Dr. W. H. BATES to an assistant professorship.

DR. F. T. H'DOUBLER has been appointed professor of mathematics at Miami University.

DR. J. E. ROWE has been appointed instructor in mathematics at Goucher College, Baltimore, Professor W. H. MALTBY having resigned to enter the profession of law.

PROFESSOR P. P. BOYD, of Hanover College, Indiana, has been granted a year's leave of absence to study at Cornell University.

AT the University of Illinois, Dr. E. B. LYTLE has been promoted to the rank of associate in mathematics; Mr. J. H. MINNICK and Mr. W. H. TAYLOR have been appointed assistants in mathematics.

At the University of North Carolina, Dr. G. R. CLEMENTS, of Williams College, Massachusetts, has been appointed professor of mathematics, and Mr. T. R. EAGLES, of Bethany College, West Virginia, has been appointed instructor in mathematics.

MR. W. J. RISLEY, of Harvard University, has been appointed professor of mathematics at Milliken University.

DR. J. B. FAUGHT, of the State Normal School, Marquette, Mich., has been appointed head of the department of mathematics at the State Normal School at Kalamazoo, Mich.

PROFESSOR F. H. HODGE, of Parsons College, Fairfield, Ia., has been appointed professor of mathematics at Franklin College, Franklin, Ind.

At the Agricultural College of Kansas, Mr. E. BARTHLOW and Mr. C. H. CLEVINGER have been appointed assistants in mathematics.

MISS GRACE CAMPBELL has been appointed instructor in mathematics at the Oregon Agricultural College.

PROFESSOR G. A. BLISS, of the University of Chicago, is in Japan on leave of absence. He will spend the winter months at the University of Paris.

PROFESSOR EUGENE ROUCHÉ, examiner of the Ecole Polytechnique of Paris, died August 19th, at the age of 78 years.

CATALOGUES of second-hand mathematical books: A. Hermann et Fils, 6 rue de la Sorbonne, Paris, catalogue 104, about 4,000 titles.—Max Weg, Königstrasse 3, Leipzig, catalogue 125, 1,600 titles in exact sciences.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

- BALDUS (R.). Ueber Strahlensysteme, welche unendlich viele Regelflächen 2ten Grades enthalten. (Diss.) Erlangen, 1910. 8vo. 46 pp.
- BIEBERBACH (L.). Zur Theorie der automorphen Funktionen. (Diss.) Göttingen, 1910. 8vo. 42 pp.
- BORHM (K.). Elliptische Funktionen. Teil II: Theorie der elliptischen Integrale; Umkehrprobleme. (Sammlung Schubert, LXI). Leipzig, Göschen, 1910. 8vo. 7 + 180 pp. Cloth. M. 5.00
- DANNACHER (S.). Die geometrischen Grundlagen der freien Perspektive. (Progr.) Frauenteld, 1910. 4to. 34 pp.
- DEMARTRES. Cours de calcul différentiel et intégral. Lille, Jauny, 1909. 8vo. 216 pp.
- FREDHOLM (I.). See MITTAG-LEFFLER (G.).
- JUNKER (F.). Repetitorium und Aufgabensammlung zur Integralrechnung. 2te verbesserte Auflage. (Sammlung Göschen, 147.) Leipzig, Göschen, 1910. 12mo. 135 pp. Cloth. M. 0.80
- LESSER (O.). See SCHWAB (K.).
- LORENSTEIN (K.). Ueber den Satz, dass eine ebene, algebraische Kurve 6ter Ordnung mit 11 sich einander ausschliessenden Ovalen nicht existiert. (Diss.) Göttingen, 1910. 8vo. 37 pp.
- MITTAG-LEFFLER (G.) et FREDHOLM (I.). Compte rendu du congrès des mathématiciens tenu à Stockholm 22-25. IX. 1909. Leipzig, Teubner, 1910. 8vo. 137 pp. Cloth. M. 4.80
- PASCAL (E.). Repetitorium der höheren Mathematik. 2te völlig ungearbeitete Auflage der deutschen Ausgabe, herausgegeben von H. E. Timerding und P. Epstein. Band I, erste Hälfte: Algebra, Differential- und Integralrechnung. Leipzig, Teubner, 1910. 8vo. 15 + 527 pp. M. 10.00
- SCHWAB (K.) und LESSER (O.). Mathematisches Unterrichtswerk zum Gebrauch an höheren Lehranstalten. Im Sinne der Meraner Lehrpläne bearbeitet. Band III, Teil I: Lehr- und Uebungsbuch für den Unterricht in der synthetischen Geometrie der Kegelschnitte und der analytischen Geometrie. Leipzig, Freytag, 1910. 8vo. 208 pp. Cloth. M. 3.00
- STAUDE (O.). Analytische Geometrie des Punktpaares, des Kegelschnittes und der Fläche 2ter Ordnung. Teil II. Leipzig, Teubner, 1910. 8vo. 451 pp. Cloth. M. 18.00

II. ELEMENTARY MATHEMATICS.

- ANGULO Y MORALES (J.). Cálculo elemental de probabilidades. Madrid, Batés, 1910. 8vo. 62 pp. P. 2.00
- FRANCHI (M. de). Elementi d'algebra, ad uso dei licei. Vol. I, per la liceale. Palermo, Sandron, 1910. 16mo. 109 pp. L. 1.50
- . Elementi d'algebra ad uso delle scuole tecniche. Palermo, Sandron, 1910. 16mo. 65 pp. L. 1.00

- GREVY (A.). Géométrie plane, à l'usage des élèves des classes de second C et D. Paris, Vuibert et Nony, 1910. 8vo. 296 pp. Fr. 3.00
- MANDL (M.). Lehrbuch der Geometrie für die oberen Klassen der Gymnasien und Realgymnasien. (IV-VIII Klasse). Wien, Manz, 1910. 8vo. 4 + 364 pp. Cloth. M. 3 80
- PERRY (J.). Practical mathematics. London, Wyman, 1910. 8vo. Cloth. 9d.
- SCHWERING (K.). Sammlung von Aufgaben aus der Arithmetik für höhere Lehranstalten. I. Lehrgang. 3te, verbesserte Auflage. Freiburg, Herder, 1910. 8vo. 7 + 59 pp. Cloth. M. 1.40
- SIBIRANI (F.). Elementi di algebra ad uso delle scuole secondarie inferiori. 2a edizione. Firenze, Le Monnier, 1910. 16mo. 145 pp. L. 1.50
- SUPPANTSCHITSCH (B.). Leitfaden der darstellenden Geometrie für die V. und VI. Klasse der Realgymnasien. Mit 212 Figuren im Text und 204 Aufgaben. Wien, Tempsky, 1910. 8vo. 71 pp. Cloth. M. 1.50
- Lehrbuch der Geometrie für Gymnasien und Realgymnasien. Mittelstufe. Planimetrie und Stereometrie. Mit 349 Figuren im Text und 1296 Fragen und Aufgaben. Wien, Tempsky, 1910. Cloth. M. 4.50
- Mathematisches Unterrichtswerk, Grundriss der Geometrie. 2te Hälfte. Wien, Tempsky, 1910. 8vo. 92 pp. Cloth. M. 1.70
- VORKAMP (G.). Mathematische Formelsammlung. Zammengestellt und herausgegeben für Schule und Praxis. Leipzig, Fritzsche und Schmidt, 1910. 12mo. 48 pp. M. 0.80
- WEITZEL (C. G.). Unterrichtsbriefe für die Buchstabenrechnung und Algebra sowie für ebene Geometrie (Planimetrie) und Anwendung der Algebra auf Geometrie. In Gesprächsform zum Selbstunterrichte verfasst. Band I. Wien, Hartleben, 1910. 8vo. 8 + 480 pp. Cloth. M. 8.75
- ZECH (J.). Tafeln der Additions- und Subtractions-Logarithmen für 7 Stellen. 4te Auflage. Berlin, Weidmann, 1910. 8vo. 203 pp. M. 3.00

III. APPLIED MATHEMATICS.

BAER. See CHRISTMANN.

- BOURLET (C.). Elements de statique graphique. Paris, Hachette, 1910. 8vo. 162 pp. Fr. 7.50
- CHRISTMANN and BAER. Grundzüge der Kinematik. Berlin, Springer, 1910. 8vo. 8 + 131 pp. Cloth. M. 5.80
- FANO (G.). Lezioni di geometria descrittiva, date nel r. politecnico di Torino. Torino-Paravia, 1910. 8vo. 19 + 426 pp. L. 16.00
- HAREN (R.). Aufgabensammlung zur Festigkeitslehre mit Lösungen (Sammlung Götschen, 491). Leipzig, Götschen, 1910. 12mo. 89 pp. Cloth. M. 0 80
- HIRSCHFELD (C. F.). Engineering thermodynamics. 2d edition. New York, Van Nostrand, 1910. 16mo. 157 pp. Boards. \$0.50
- HOPKINSON (B.). Vibrations of systems having one degree of freedom. New York, Putnam, 1910. 8vo. 54 pp. \$0.75

MATRICULATION mechanics papers from 1898-1900 and from September 1902 to June 1910 with model solutions to the last paper and additional questions. London, Clive, 1910. 8vo. 1s. 6d.

MÜLLER (E.). Technische Uebungsaufgaben für darstellende Geometrie. Wien, Deuticke, 1910. 8vo. M. 2.50

RÖTTINGER (M.). See WALTHER (K.).

WALTHER (K.) und RÖTTINGER (M.). Technische Wärmelehre (Thermodynamik). (Sammlung Göschen, 242.) Leipzig, Göschen, 1910. 12mo. 144 pp. Cloth. M. 0.80

WELLSCH (S.). Theorie und Praxis der Ausgleichsrechnung. Band II. Probleme der Ausgleichsrechnung. Wien, Fromme, 1910. 8vo. 9 + 217 pp. M. 7.50

THE SEPTEMBER MEETING OF THE SAN
FRANCISCO SECTION.

THE eighteenth regular meeting of the San Francisco Section of the American Mathematical Society was held at the University of California on Saturday, September 24, 1910. The following members were present:

Professor R. E. Allardice, Mr. B. A. Bernstein, Professor H. F. Blichfeldt, Professor G. C. Edwards, Professor R. L. Green, Professor M. W. Haskell, Professor L. M. Hoskins, Professor D. N. Lehmer, Professor H. C. Moreno, Professor C. A. Noble, Professor T. M. Putnam.

Professor Blichfeldt occupied the chair. The following officers were elected for the ensuing year: Chairman, Professor Lehmer; Secretary, Professor Moreno; program committee, Professors Blichfeldt, Haskell, and Moreno.

The dates of the next two meetings were fixed as April 8, 1911, and October 28, 1911.

The following papers were read at this meeting:

(1) Professor H. F. BLICHFELDT: "On the order of linear homogeneous groups. IV."

(2) Professor D. N. LEHMER: "On the study of the general cubic by means of three involutions of rays in the plane."

Abstracts of the papers are given below:

1. In the theorems by Professor Blichfeldt on groups of linear homogeneous substitutions of determinant unity, a group of special nature is of importance, called a "self-conjugate subgroup H ." It is contained in a linear group G when certain conditions are fulfilled. The theorems referred to suffer a lack of completeness due to the fact that it has not been proved that H actually contains fewer substitutions than G . In the present paper this proof has been supplied under the condition that G be primitive; besides, it is found that the number of variables n is divisible by the prime p associated with H in the theorems. An immediate result is the lowering of the known superior limit to the order of G , particularly when n is a prime.

2. Given three unrelated involutions of rays in the plane, with centers at A , B , and C , then for any point P in the plane the rays at A , B , and C that correspond in the involutions at those points to the rays PA , PB , and PC will not generally meet in a point. If, however, they do meet in a point P' , then the points P and P' are said to be conjugate with respect to all three involutions. The locus of points that have conjugate points with respect to three involutions is found to be the general plane cubic. Professor Lehmer studies the curve from this point of view and connects the theory with the theory of the curve as developed by Schroeter in his *Ebene Kurven dritter Ordnung* (Leipzig, 1888).

C. A. NOBLE,
Secretary of the Section.

A NEW PROOF OF THE THEOREM OF WEIERSTRASS CONCERNING THE FACTORIZATION OF A POWER SERIES.

BY DR. W. D. MACMILLAN.

IN the BULLETIN for April, 1910, Bliss gives a simplified proof of the following theorem due to Weierstrass:

Let $f(y; x_1, \dots, x_p)$ be a convergent power series in y and $x_1, \dots, x_p$, such that $f(y; 0, \dots, 0)$ begins with a term of degree n . Then $f(y; x_1, \dots, x_p)$ is factorable in the form

$$f(y; x_1, \dots, x_p) = [y^n + a_1 y^{n-1} + \dots + a_n] \cdot g(y; x_1, \dots, x_p),$$

where $a_1, \dots, a_n$ are convergent power series in $x_1, \dots, x_p$ vanishing for $x_1 = x_2 = \dots = x_p = 0$, and g is a convergent power series in $y; x_1, \dots, x_p$ which has a constant term different from zero.

Since $g(y; x_1, \dots, x_p)$ has a constant term, we may denote its reciprocal by $\phi(y; x_1, \dots, x_p)$ and state the theorem in the following form:

Let $f(y; x_1, \dots, x_p)$ be a convergent power series in $y; x_1, \dots, x_p$ such that $f(y; 0, \dots, 0)$ begins with a term of degree n . Then a convergent power series $\phi(y; x_1, \dots, x_p)$, having a constant term different from zero can be found such that the product

$f(y; x_1, \dots, x_p) \cdot \phi(y; x_1, \dots, x_p) = p^{(n)} = y^n + a_1 y^{n-1} + \dots + a_n$ is a polynomial in y of degree n and $a_1, \dots, a_n$ are convergent power series in $x_1, \dots, x_p$ vanishing for $x_1 = x_2 = \dots = x_p = 0$.

For the purpose of determining the coefficients of the series ϕ it will simplify the notation if we put $x_i = \xi_i \mu$. Then the series f , ϕ and $p^{(n)}$ can be written

$$\begin{aligned} f &= -y^* [1 - b_0] + b_1 \mu + b_2 \mu^2 + \dots, \\ \phi &= c_0 + c_1 \mu + c_2 \mu^2 + \dots, \\ p^{(n)} &= -y^* + p_1 \mu + p_2 \mu^2 + \dots, \end{aligned} \quad (1)$$

where the b_k are known power series in y and are homogeneous in the ξ_i of degree k , b_0 containing terms in y only and vanishing for y equal to zero. The c_k are power series in y whose coefficients are to be determined, and the p_k are polynomials in y , also to be determined, of degree $n - 1$.

On taking the product of f and ϕ we find

[illegible]

Comparing the coefficients of the various powers of μ we have

[illegible]

These equations can be solved successively. From the first we have at once $c_0 = 1/(1 - b_0)$. From the second it is seen that, since b_1 and c_0 are known power series in y , p_1 can be chosen uniquely so that $b_1 c_0 - p_1$ shall be divisible by y^n . If we suppose p_1 so chosen, we can then write $b_1 c_0 - p_1 = \beta_1 y^n$ and the solution for c_1 is $c_1 = \beta_1/(1 - b_0)$. It is clear that c_1 and p_1

are homogeneous of degree 1 in the ξ_i . The third equation can be solved in the same manner, the polynomial p_2 being chosen so that the right member is divisible by y^n . Since this is possible in one and only one way, the polynomial p_2 and the power series c_2 are uniquely determined. If we suppose $c_1, \dots, c_{k-1}$ to have been computed, then p_k can be determined uniquely so that

$$\sum_{j=1}^k b_j c_{k-j} - p_k$$

shall be divisible by y^n . After removing the factor y^n , c_k is determined by dividing the result by $1 - b_0$. One can thus compute as many coefficients as is desired, all the coefficients of a given order in μ being obtained in a single operation.

The proof of convergence is by dominant functions. Let us suppose that the series f converges for $|y| = 1/\sigma$, $|x_i| = 1/\rho_i$, and let us set

$$X = \frac{1}{(1 - \rho_1 x_1)(1 - \rho_2 x_2) \cdots (1 - \rho_p x_p)} - 1.$$

Then a positive quantity M can be chosen so large that the series

$$F = -y^n + M \frac{\sigma^{n+1} y^{n+1}}{1 - \sigma y} + M \frac{X}{1 - \sigma y}$$

shall dominate the series f . By the above process a power series $\Phi(y; x_1, \dots, x_p)$ and a polynomial

$$P^{(n)} = -y^n + A_1 y^{n-1} + \cdots + A_n$$

can be found such that the coefficients of Φ are all positive and greater than the moduli of the corresponding coefficients of the series ϕ ; and the coefficients of the A_k , which are power series in $x_1, \dots, x_p$ vanishing with these quantities, are all positive and greater than the moduli of the corresponding coefficients of the series a_k . Hence if the series Φ and the A_k converge so also do the series ϕ and the a_k .

Let us take

$$\Phi = \frac{1 - \sigma y}{\omega - sy},$$

where $s = \sigma + M\sigma^{n+1}$, and ω is as yet undetermined. Then

the equation $F \cdot \Phi = P^{(n)}$ becomes

$$\left[-y^n + M \frac{\sigma^{n+1} y^{n+1}}{1 - \sigma y} + M \frac{X}{1 - \sigma y} \right] \frac{1 - \sigma y}{\omega - \sigma y} = -y^n + A_1 y^{n-1} + \dots + A_n,$$

which reduces readily to

$$(4) \quad \frac{(1 - \omega)}{s} \left[\frac{y^n - \frac{MX}{1 - \omega}}{\frac{\omega}{y - \frac{\omega}{s}}} \right] = A_1 y^{n-1} + \dots + A_n.$$

The left member of equation (4) will be a polynomial in y of degree $n - 1$ provided ω satisfies the relation

$$(5) \quad \frac{MX}{1 - \omega} = \left(\frac{\omega}{s} \right)^n.$$

On putting $\omega = 1 - z$ we get

$$(6) \quad z = \frac{s^n MX}{(1 - z)^n}.$$

Thus z is expansible as a power series in X the coefficients of which are all positive, and furthermore z vanishes with X . Hence

$$(7) \quad \Phi = \frac{1 - \sigma y}{\omega - \sigma y} = \frac{\sigma}{s} \left[1 + \frac{M\sigma^n + z}{1 - z - \sigma y} \right]$$

is expansible as a power series in $y, x_1, \dots, x_p$, the coefficients of which are all positive. The polynomial $P^{(n)}$ becomes

$$\begin{aligned} P^{(n)} = -y^n + \frac{1 - \omega}{s} \left[\frac{y^n - \left(\frac{\omega}{s} \right)^n}{y - \left(\frac{\omega}{s} \right)} \right] &= -y^n + \frac{z}{s} y^{n-1} \\ &+ \frac{z(1 - z)}{s^2} y^{n-2} + \dots + \frac{z(1 - z)^{n-1}}{s^n}. \end{aligned}$$

From equation (6) it is seen that the coefficients of this polynomial,

$$\frac{z(1-z)^k}{s^{k+1}} = \frac{s^{n-k-1}MX}{(1-z)^{n-k}} \quad (k = 1, \dots, n-1),$$

are expansible as power series in $x_1, \dots, x_p$, with positive coefficients, vanishing for $x_1 = \dots = x_p = 0$. Hence the theorem is proved.

UNIVERSITY OF CHICAGO,
August 14, 1910.

KOWALEWSKI'S DETERMINANTS.

Einführung in die Determinantentheorie einschliesslich der unendlichen und der Fredholmschen Determinanten. By GERHARD KOWALEWSKI. Leipzig, Veit & Comp., 1909. 540 pp.

A PRELIMINARY survey of the contents of this book may best be obtained by dividing it into three nearly equal parts. The first of these parts comprises Chapters I–X and deals with the pure theory together with the single application to systems of linear equations*—a subject both historically and logically so intimately connected with the theory as to be almost inseparable from it. In the second part—Chapters XI–XVI—certain applications to algebraic, analytic, and geometric problems are treated. The third part consists of Chapters XVII–XIX and deals with two extensions of the idea of determinants obtained by allowing the order of the determinant to increase indefinitely. Both the theory of these determinants (infinite determinants and Fredholm determinants) and the closely related theory of the corresponding generalizations of systems of linear equations (linear equations with an infinite number of variables and linear integral equations) are treated in these chapters. While the introduction of such subjects into a book on determinants is not wholly unprecedented—Mathews' revision of Scott's book containing a brief discussion of infinite determinants—it must still, in view of its extent, be regarded as an innovation. This feature of the book is to be welcomed and will doubtless be imitated by other writers of text-books on this subject.

* A few equations of higher degree (secular equation, etc.) which are intimately connected with the theory of determinants are also considered in this first part.

We will now take up the chapters briefly one by one :

Chapter I (5 pages) is an interesting historical introduction in which translations are given of a passage in a letter of Leibnitz containing a definition of determinants, and of the note by Cramer where his rediscovery of determinants and his rule for solving linear equations were first published.

Chapter II (17 pages) gives, after a rather elaborate preparation for the specification for the signs of the terms of a determinant, the general definition with applications to two and three rowed determinants.

In Chapter III (9 pages) the very simplest properties of determinants (interchange of two rows, multiplication of the elements of a row by a constant, etc.) are deduced ; and at the close it is shown how certain of these properties characterize the determinant completely. This fact is not used in later chapters, however, as Kronecker has shown that it can be used.

Chapter IV (13 pages) is entitled Sub-Determinants and contains Laplace's development with the development according to the elements of a row as a special case. It closes with a treatment of Vandermonde's determinant, that is the determinant of the n th order whose i th row is

$$a_i^{n-1}, a_i^{n-2}, \dots, a_i, 1.$$

In Chapter V (19 pages) the theory of linear algebraic equations is treated. It begins with a proof of Cramer's Rule. Then after the conception of rank has been introduced and a few simple theorems concerning it have been proved, the subject of linear dependence is discussed. Then systems of *homogeneous* equations are taken up at length ; and finally the non-homogeneous case is treated.

Chapter VI (13 pages) is entitled Multiplication of Matrices and Determinants. Here, and constantly from this point on, the following notation and terminology are used : If

$$x_1, x_2, \dots, x_n, \quad y_1, y_2, \dots, y_n$$

are two systems of n quantities each, the quantity

$$x_1 y_1 + x_2 y_2 + \dots + x_n y_n$$

is called their inner product (or simply their product) and denoted by (xy) . This introduction of an idea of Grassmann

is a distinctly good thing. It is, however, to be regretted that either here or at some other early stage of the book the subject were not treated in a somewhat more extensive manner by introducing the term *vector in space of n dimensions*, the conception of the sum of two vectors, the norm of a vector, the orthogonality of two vectors, etc. All of these conceptions are used freely in Chapter XVII in the case of space of an infinite number of dimensions, but before that point, where we have to deal with space of a finite number of dimensions, the term vector is not used at all, and such of these conceptions as are introduced occur in the midst of special investigations, for instance orthogonality is first explicitly mentioned on page 276 in an investigation concerning quadratic forms, although the idea (and also the conception but not the term *norm*) was really used in Chapter X. This is an illustration of the lack of complete coördination between the different parts of the book to which we shall have occasion to return later.

As a generalization and at the same time an application of this conception of the inner product we next have the product of two matrices

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{m1} & \cdots & a_{mn} \end{vmatrix}, \quad \begin{vmatrix} b_{11} & \cdots & b_{1n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ b_{m1} & \cdots & b_{mn} \end{vmatrix},$$

which is defined as the *determinant* of the m th order whose general element (i th row and j th column) is the product, in the sense just explained, of the i th row of the first matrix into the j th row of the second. It is proved that the product, in this sense, of two square matrices is equal to the product of their determinants (so that the theorem of multiplication of determinants is established), that when $m > n$ the product is zero, and that when $m < n$ it is the sum of the products of all corresponding m -rowed determinants in the two matrices. As an elegant application for the law of multiplication of determinants a formula for the square of Vandermonde's determinant is given which is essentially the expression for the discriminant of an algebraic equation in terms of sums of powers of the roots.

Chapter VII (33 pages) is entitled Determinants whose Elements are Minors of Another. It opens with the familiar theorems about the determinant whose elements are the cofactors

of another, and concerning the minors of this determinant. Some theorems of Sylvester of a more complicated character follow; and a treatment of bordered determinants is also included.

Chapter VIII (21 pages) on Symmetrical Determinants contains, besides a discussion of several special symmetrical determinants, an interesting proof that a symmetrical determinant of rank r contains at least one non-vanishing principal minor of order r , and a treatment of the secular equation, that is the equation obtained by equating to zero a determinant derived from a symmetrical determinant with real constant elements by adding the variable x to each element of the principal diagonal. The last section refers to a generalization of this equation which bears the same relation to a Hermitian matrix (that is to a matrix in which a_{ij} and a_{ji} are conjugate imaginary quantities) that the secular equation bears to real symmetric matrices. Some properties of Hermitian matrices are also briefly established, but the name *Hermitian*, which is freely used later in the book, is not mentioned.

In Chapter IX (27 pages) on Skew Symmetrical Determinants the following two fundamental theorems are first proved: *first* that a skew symmetric determinant of odd order is always zero, and *secondly* that a skew symmetric determinant of even order is the square of an integral rational function of its elements. This integral rational function, known as the Pfaffian, is studied, and its close analogy in theory and application to a determinant is brought out. The rank of a skew symmetric determinant is then considered; and finally a number of special determinants connected with this subject are treated.

Chapter X (19 pages) is on Orthogonal Determinants. After the fundamental properties of these determinants have been established, some special theorems of Brioschi, Siacci, and Stieltjes are proved, among which the first theorem of Siacci is especially noteworthy both for itself and for the application made of it to a simple proof of Cayley's formulæ. These formulæ, by means of which the elements of an orthogonal determinant are expressed rationally by means of $\frac{1}{2}n(n-1)$ parameters, are here proved on the supposition that the determinant has the value $+1$ and that the determinant obtained by adding 1 to each element of the principal diagonal is not zero. The formulæ are worked out in detail for determinants of the second and third orders.

A footnote at the beginning of this chapter tells us that the

study of orthogonal determinants shall here be restricted to the case in which all the elements are real. This seems at first quite incomprehensible since, except at one or two points, where the restriction might easily have been explicitly made, the results and methods of the chapter apply equally well to the case of complex elements. The reason for the restriction to real elements becomes evident only at a much later point in the book (page 282) where, in the course of a special investigation on Hermitian forms, the word *orthogonal* is applied to the case of complex elements when the quantity

$$a_{i1}\bar{a}_{j1} + a_{i2}\bar{a}_{j2} + \cdots + a_{in}\bar{a}_{jn}$$

has the value 1 or 0 according as i and j are equal to or different from each other, — the a 's being the elements and dashes being used to denote the conjugate complex quantity. While such a departure from well established usage in terminology can hardly be countenanced, it is certain that we have here a conception of importance* which under some other name is well worthy of place in a book on determinants. It is only to be regretted that it was not treated in a somewhat systematic manner in the chapter on orthogonal determinants.

We pass now to what we have called the second part of the book, — Applications.

Chapter XI (32 pages) on Resultants and Discriminants deals exclusively with binary forms. Sylvester's dialytic method (supplemented so as to be complete), Bézout's method, the relation of resultant and discriminant to the linear factors of the forms, finally their invariant character together with a few words about invariants of binary forms in general are some of the main subjects treated. The treatment at some points, for instance in §§ 78, 79, is remarkably elegant and lucid.

Chapter XII (43 pages) on Linear and Quadratic Forms opens with a consideration of the rank of a system of linear forms and, in case the number of forms is equal to the number of variables, their resultant regarded as invariants with regard to linear transformations. Then comes a brief treatment of bilinear forms in which, for the first time, that product of two matrices is considered which is itself a matrix. The remainder

* Cf. Frobenius, *Crelle*, vol. 95 (1883), p. 267; E. Study, *Math. Ann.*, vol. 60 (1905), p. 321, and I. Schur, *Math. Ann.*, vol. 66 (1909), p. 489, where references to some further literature will be found.

of the chapter is devoted to quadratic forms, their reduction to sums of squares, the law of inertia and the classification of real quadratic forms, the adjoint form, the rational invariants of pairs of quadratic forms, etc. Here, with slight exceptions, both the facts proved and the methods of proof are identical with those contained in the reviewer's Introduction to Higher Algebra.

Chapter XIII (36 pages) is on Elementary Divisors, and gives, in very brief space, the fundamental theorems on the equivalence of pairs of bilinear or quadratic forms. The method used, which is due to Stickelberger, depends, as did Weierstrass's original method, on a reduction to normal form, and like that method it involves unnecessary irrationalities. It depends on analytic tricks for which no motive is apparent, though the author has met this last objection to some slight extent by a similar, though much simpler discussion for a single quadratic form in § 106. Its compensating advantage, apart from its brevity, is that, like Weierstrass's original method, it gives a practicable method of approach to the subject of real quadratic forms, a subject in which, however, only a very special case is here considered.

The last part of this chapter contains an interesting exposition of the real *orthogonal* reduction of a single real quadratic form to a form in which only square terms enter, together with a generalization to the case of Hermitian forms. This work does not involve the theory of elementary divisors in any way, so that the place which has been given it, at the end of the chapter on elementary divisors, is unfortunate as many readers without the time or inclination to make themselves familiar with this theory will be likely to miss altogether this elegant treatment.

Chapter XIV (30 pages) on Functional Determinants opens, after the conception of the Jacobian has been defined, with the remark that the Jacobian

$$\frac{d(u_1, \dots, u_n)}{d(x_1, \dots, x_n)}$$

can be written as the quotient

$$\left| \begin{array}{cccc} d_1 u_1 & d_1 u_2 & \dots & d_1 u_n \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ d_n u_1 & d_n u_2 & \dots & d_n u_n \end{array} \right| \div \left| \begin{array}{cccc} d_1 x_1 & d_1 x_2 & \dots & d_1 x_n \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ d_n x_1 & d_n x_2 & \dots & d_n x_n \end{array} \right|$$

where $d_i u_j$ denotes the differential of u_j which corresponds to the increments

$$d_i x_1, d_i x_2, \dots, d_i x_n,$$

and where the increments of the x 's are chosen arbitrarily but so that their determinant is not zero. It is then shown that, under suitable restrictions, the Jacobian can be obtained as the limit of a similar ratio where the differentials of the u 's are replaced by increments. It would have been interesting for the reader to have been told whether this result is of any importance, and if so why.

There follows the law of multiplication for functional matrices and determinants. The transformation

$$\xi_1 = u_1(x_1, \dots, x_n),$$

$$\xi_2 = u_2(x_1, \dots, x_n),$$

$$\begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \cdot \cdot \cdot \cdot \cdot \cdot \cdot$$

$$\xi_n = u_n(x_1, \dots, x_n),$$

when the Jacobian of the u 's is not zero, is then studied for the neighborhood of individual points, the work being carried through in detail for the case $n = 2$. Then comes a treatment of the effect of the rank of the functional matrix on the number of independent functions in the neighborhood of a non-singular point. Finally inverse functions, implicit functions, and the minors of the Jacobian are taken up.

Chapter XV (17 pages) on Wronskian and Gramian Determinants is one of the most novel and timely in the book. If $f_1(x), \dots, f_n(x)$ are continuous (or more generally if these functions and their squares are integrable), but not necessarily real, throughout the interval $a \leq x \leq b$, we understand by their Gramian the determinant of the n th order in which the term in the i th row and j th column is

$$\int_a^b f_i(x) \overline{f_j(x)} dx,$$

the dash indicating the conjugate complex quantity. The vanishing of this determinant is a necessary and sufficient condition

that $f_1, \dots, f_n$ be linearly dependent.* If $f_1, \dots, f_n$ are linearly independent, their Gramian is always positive. On the other hand, if

$$a_{i1}, a_{i2}, \dots, a_{ik} \quad (i = 1, 2, \dots, n)$$

are n sets of k constants each, we understand by their Gramian the determinant of the n th order in which the element in the i th row and j th column is

$$a_{i1}\bar{a}_{j1} + a_{i2}\bar{a}_{j2} + \dots + a_{ik}\bar{a}_{jk}.$$

Here again the vanishing of this determinant is a necessary and sufficient condition for the linear dependence of the sets of constants; and if these sets are linearly independent, their Gramian is positive. We have here a striking analogy, but unfortunately no attempt is made in the book under review to explain *why* this analogy exists. Such an explanation would not have been difficult, for instance one may regard the case last mentioned as a special case of the earlier one by letting $a = 0$, $b = k$ and

$$f_i(x) = a_j \quad \text{when } j-1 < x < j \\ (i = 1, 2, \dots, n; j = 1, 2, \dots, k).$$

Or one may take the more general point of view of considering n sets of k functions each

$$f_{i1}(x), f_{i2}(x), \dots, f_{ik}(x) \quad (i = 1, 2, \dots, n)$$

continuous throughout the interval $a \leq x \leq b$, or, more generally, such that these functions and their squares are integrable throughout this interval. By their Gramian we then understand the determinant of the n th order in which the element in the i th row and j th column is

$$\int_a^b [f_{i1}(x)\bar{f}_{j1}(x) + \dots + f_{ik}(x)\bar{f}_{jk}(x)]dx,$$

* If $f_1, \dots, f_n$ are not assumed to be continuous, we understand here by linear dependence throughout the interval $a \leq x \leq b$ the existence of n constants $c_1, \dots, c_n$ not all zero and such that

$$c_1 f_1 + \dots + c_n f_n$$

vanishes at every point of this interval except at the points of a set of content zero.

and here again this Gramian will always be positive except when the n sets of functions are linearly dependent, in which case it vanishes. This conception clearly includes the two above mentioned as special cases.

However much the chapter under consideration is to be welcomed, it is to be regretted that it had not been made more complete. This might have been effected in part by collecting here various discussions which are scattered through the later chapters and which really refer to this subject; for instance the proof of Hadamard's inequality for the absolute value of a determinant on pages 458-460. This inequality is simply a corollary to the theorem that a Gramian (in any of the senses above mentioned) can never exceed the product of the elements of its principal diagonal.* Again the relation of Gramians to definite Hermitian forms is not considered in this chapter.

One section of this chapter is devoted to the relation of Wronskians to the subject of linear dependence, but after three and a half pages of this discussion the reader is left without the knowledge of the most important, and one of the most easily proved theorems in this subject, namely that if n functions are analytic, the identical vanishing of their Wronskian is a sufficient condition for linear dependence.

In comparing the Wronskian with the Gramian test for linear dependence the important fact is pointed out, that while the former test requires, in the case of n functions, the existence of the first $n - 1$ derivatives, for the latter the continuity of the functions is sufficient. The even more important fact that the Gramian test applies to functions in any number of variables is not mentioned.

Chapter XVI (32 pages) is entitled: Some Geometrical Applications of Determinants. More than half of it is devoted to applications in the domain of elementary analytic geometry: The area of a triangle in terms of the coördinates of its vertices and in terms of the equations of its sides; the product of the areas of two triangles in a plane in terms of the distances between their six vertices; a similar result in case the two triangles are inscribed in a circle of radius r ; the characteristic relation between the distances of four points on a circle; finally the generalization of all of these results to space. Then come two

* If none of the functions (or systems of functions) are identically zero, it reaches this upper limit when and only when the functions (or systems of functions) are mutually orthogonal.

sections on affine transformations in the plane, with special reference to the fact that all areas are changed in the same ratio. Finally the geometrical interpretation of the Jacobian is discussed of which the property of affine transformations just mentioned is a special case, the work being here again confined to the neighborhood of a non-singular point.

* * *

We pass now to the third part of the book.

Chapter XVII (86 pages): Determinants of Infinitely High Order, might well have been split into two chapters. It should at least have been clearly indicated that it consists of two parts wholly independent of each other. The first part, somewhat the shorter of the two, closes on page 407. If we have a doubly infinite array of quantities

$$\begin{array}{cccc} a_{11} & a_{12} & a_{13} & \dots \\ a_{21} & a_{22} & a_{23} & \dots \\ a_{31} & a_{32} & a_{33} & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{array}$$

and if we form from them the determinant of the n th order

$$\begin{vmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{vmatrix},$$

it may happen that this determinant has a finite limit when $n = \infty$. This limit is then called a determinant of infinitely high order and is denoted by putting vertical bars on the two sides of the infinite array above. Our author considers primarily *normal* determinants, that is determinants for which the double series

$$\sum_{i,j=1}^{\infty} c_{ij}$$

converges where

$$c_{ii} = |a_{ii} - 1|, \quad c_{ij} = |a_{ij}| \quad (i \neq j).$$

Such determinants are necessarily convergent as are also their

minors and all determinants obtained from them by replacing a finite number of their rows or columns by quantities which are in absolute value less than some constant.

If in the infinite system of linear equations with an infinite number of variables

$$a_{11}x_1 + a_{12}x_2 + \dots = b_1,$$

$$a_{21}x_1 + a_{22}x_2 + \dots = b_2,$$

$$\begin{array}{cccccccc} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

the a_v 's are such that the infinite determinant formed from them is a normal determinant, and the b 's do not exceed in absolute value some constant, it can be shown that normal determinants and their minors can be used for the solution of the system precisely as finite determinants can be used for the solution of finite systems of linear equations with a finite number of variables. This, together with the extension of numerous properties of finite determinants to infinite normal determinants, forms the contents of this first part of the chapter.

The second part contains an exposition of E. Schmidt's theory of systems of linear equations, and *does not deal with infinite determinants at all*. It is true that determinants occur here whose order n increases indefinitely, but these determinants do not in general themselves approach finite limits when $n = \infty$ but only the ratio of two such determinants.* The subject has been made more attractive by the free use of the conception and language of vectors in space of an infinite number of dimensions. One misses, however, any statement of the relation of this method of Schmidt to the method of infinite determinants explained in the earlier part of the chapter. The relations would seem to be, in part, these :

1. Both methods deal with the solution of the infinite system of equations referred to above, and neither method treats the problem in all its generality.

2. In the method of infinite determinants it is assumed that the double series $\sum c_v$ converges, while in Schmidt's method not even the convergence of the individual rows of this series,

*The determinants themselves can be made to converge by dividing the equations of the system we wish to solve by constants such as to make the series of the squares of the coefficients of the individual equations have values less than or equal to 1.

or, what amounts to the same thing, the convergence of the series

$$\sum_{j=1}^{\infty} |a_{ij}| \quad (i = 1, 2, \dots)$$

is demanded, but merely the convergence of the series

$$\sum_{j=1}^{\infty} |a_{ij}|^2 \quad (i = 1, 2, \dots).*$$

3. The method of infinite determinants yields, in the cases to which it is applicable, every solution $(x_1, x_2, \dots)$ which is such that all the x_i 's are in absolute value less than some constant, while Schmidt's method yields only those solutions $(x_1, x_2, \dots)$ for which the series $\sum |x_i|^2$ converges.

4. When we use infinite determinants, we obtain the solution of the infinite system † as the limit for $n = \infty$ of the solution of the finite system

$$a_{11}x_1 + \dots + a_{1n}x_n = b_1,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$a_{n1}x_1 + \dots + a_{nn}x_n = b_n,$$

that is we allow the number of equations and the number of unknowns to increase *simultaneously* always remaining equal. In Schmidt's method we start from the system

$$a_{11}x_1 + \dots + a_{1k}x_k = b_1,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$a_{n1}x_1 + \dots + a_{nk}x_k = b_n$$

and first, keeping n fixed, pass to the limit $k = \infty$; ‡ only after this has been done do we allow n to become infinite. In order to be able to carry this method through, new formulae must be obtained for the solution of the system of equations last written when $k > n$.

Some such explanation as this of the scope of the two methods and their relation to one another should have been given in the

* We pass over here the less easily stated conditions on the b 's.

† At least in case the determinant of the system is not zero.

‡ This first passage to a limit is, in practice, avoided by treating *directly* the case $k = \infty$.

chapter under discussion. Instead of this, the two methods are merely placed side by side without comment. Moreover in the latter part of the chapter we miss any clear statements of the results obtained, so that it is almost impossible to find out precisely what Schmidt's method accomplishes without reading the greater part of the forty odd pages here devoted to it.

Chapter XVIII (50 pages) on Linear Integral Equations opens with a discussion of Fredholm's determinant. This, like the infinite determinant, is the limit of a determinant of the n th order as n becomes infinite; but while, in the case of the infinite determinant, the element in the i th row and j th column of the determinant of the n th order does not change with n when once n is greater than both i and j (just as in the case of an infinite series, which is the limit of the sum of n terms, these terms do not change with n), in the case of Fredholm's determinant the elements approach zero, or, in the case of the elements of the principal diagonal, one, as n becomes infinite (just as in the case of a definite integral, which is also the limit of a sum of n terms, each term approaches zero as n becomes infinite). To make this rough statement more precise, consider a function $f(x, y)$ continuous throughout the square

$$S \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1.$$

Divide S into n^2 equal squares by lines parallel to the axes of x and y and denote the coordinates of the centres of these squares by

$$(x_i, y_j) \quad (i, j = 1, 2, \dots, n),$$

where the notation is such that both x_i and y_j increase as the subscript increases. Then, as n becomes infinite, the determinant

$$D_n = \begin{vmatrix} 1 + \frac{f(x_1, y_1)}{n} & \frac{f(x_1, y_2)}{n} & \dots & \frac{f(x_1, y_n)}{n} \\ \frac{f(x_2, y_1)}{n} & 1 + \frac{f(x_2, y_2)}{n} & \dots & \frac{f(x_2, y_n)}{n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{f(x_n, y_1)}{n} & \frac{f(x_n, y_2)}{n} & \dots & 1 + \frac{f(x_n, y_n)}{n} \end{vmatrix}$$

approaches a finite limit called the Fredholm determinant of the function f for the region S .

About half of the chapter under consideration is devoted to the theory of these Fredholm determinants; their minors and relations between them, the product of two Fredholm determinants, etc., being discussed. The *results* are due to Fredholm; the *method*, that of passing to the limit from the algebraic case, was used by Fredholm as a heuristic one for discovering the facts. His method of proof, however, was a different and more elegant one. As a rigorous method of proof, this method was first used by Hilbert for establishing some of Fredholm's results, along with certain other results with which we are not here concerned. In other cases (the minors of higher orders) the chapter now under consideration seems to give the first exposition of this method of proof.

The other half of the chapter is devoted to the application of Fredholm's determinants to the integral equation of the second kind

$$\phi(x) + \int_0^1 f(x, y)\phi(y)dy = \psi(x),$$

where f and ψ are known functions, ϕ the function to be determined. Here the method used is that of Fredholm not of Hilbert, that is the integral equation is treated directly, not as the limit of a system of linear algebraic equations. The analogy with the case of linear algebraic equations is frequently mentioned and sometimes elaborated at some length, but we fancy this analogy will seem pretty obscure unless the reader is fortunate enough to notice the note on page 544, where the method of obtaining the integral equation as the limit of the algebraic system is explained and falsely attributed to Hilbert. It was known to Fredholm, and had been explained by Volterra in 1896.

Here, as in Chapter XV, it is unfortunate that no mention is made of the fact that methods and results apply without change to functions in any number of variables.

Chapter XIX (35 pages) is entitled: Hilbert's Characteristic Functions of a Real Symmetric Kernel, and gives, with certain interesting modifications of detail, some of the more important parts of E. Schmidt's dissertation (*Mathematische Annalen*, volume 63). It deals with the above written integral equation, chiefly in the case in which f is a symmetric function of (x, y) , with special attention to the case in which ψ vanishes identically. While these developments of Schmidt are of the great-

est importance and elegance, it is hard to see why they should be inserted into a book on determinants. More germane would seem to be Hilbert's theory of quadratic forms with an infinite number of variables, an exposition of which would have been, for other reasons also, most desirable.

* * *

Properly regarded, the theory of determinants is hardly more than a shadow of the theory of matrices, and just so far as one attempts to ignore this fact does the subject become an artificial one or, at best, a tool for other investigations. Every book on determinants must be dominated more or less by the conception of the matrix, which first appears as an inanimate object, a mere rectangular array of quantities from which determinants, whose vanishing or non-vanishing are observed or whose values are used, are cut out. Later, however, the matrix appears as an animated being capable of combination by addition and at least two kinds of multiplication with its kind, and it is in this aspect that the conception becomes a highly fruitful one. In the book under review the matrix as an inanimate object is freely used, and the less important kind of multiplication, according to which the product of two matrices is a determinant (cf. what was said in connection with Chapter VI) is adequately treated. That two matrices can be added together or that a matrix can be multiplied by an ordinary quantity (scalar) seems to be nowhere even mentioned. The extremely important kind of multiplication according to which the product of two square matrices of the n th order is another matrix of the n th order is not taken up until the section on Bilinear Forms in Chapter XII, that is, in what we have called the second part of the book, after the *theory* of determinants of finite order has been completed. Thus it will be seen that the idea of the matrix is used somewhat reluctantly. That the book nevertheless remains a good book is due largely to the fact that in many places where the word *determinant* is used a matrix is meant. A determinant is quite properly defined (page 19, near the top) as a *quantity* obtained by combining by the well-known rule the elements of a square matrix. On page 24 we have the theorem that a determinant is unchanged by an interchange of rows with columns. Later (page 161) comes the theorem that an orthogonal determinant remains orthogonal if rows are interchanged with columns. What a triviality if the earlier statement that every determinant remains unchanged was true!

The fact of course is that the theorem about orthogonal determinants is meant to refer not to a determinant, but to an orthogonal matrix. Indeed the whole chapter on orthogonal determinants is really a chapter on orthogonal matrices; the only fact proved, and the only non-trivial fact which can be proved about an orthogonal determinant (i. e., the determinant of an orthogonal matrix) being that it is equal to $+1$ or -1 . All that we can mean by saying that a determinant (i. e., a quantity) is orthogonal is that an orthogonal matrix exists of which it is the determinant; and this is clearly true when and only when the determinant is $+1$ or -1 . Consequently, to take one more example, the theorem that the product of two orthogonal determinants is an orthogonal determinant is a triviality—it merely says that if $a = \pm 1$ and $b = \pm 1$, then $ab = \pm 1$. What is *meant* is, of course, that the product of two orthogonal *matrices* is an orthogonal *matrix*, the kind of product meant being that which is defined only fifty pages later.

The advantage of the free use of the word and conception *matrix* would be not merely to avoid calling two different things by the same name, perhaps even in a single sentence,—a most pernicious thing in mathematics—but also to permit much briefer and more transparent formulations of many proofs. For instance, it follows at once from the definition that a matrix A is orthogonal when and only when

$$A'A = I,$$

where A' is the conjugate (transposed) of A and I is the unit matrix

$$I = \begin{vmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 1 \end{vmatrix}.$$

It follows that

$$A' = A^{-1}$$

and this is merely a more compact and elegant way of stating Theorem 43 on page 160. Since a matrix is always commuta-

tive with its reciprocal, we see that

$$AA' = I$$

and consequently A' is orthogonal. (Theorem 44, page 161.)

If A and B are both orthogonal

$$A' = A^{-1}, \quad BB' = I.$$

Multiplying these equations together gives

$$A'BB' = A^{-1}.$$

Consequently

$$A'BB'A = I.$$

But, $B'A$ being the conjugate of $A'B$, the orthogonality of $A'B$ is thus proved. (Theorem 45, page 161.)

The introduction of proofs of this character would presuppose a treatment of the fundamental facts in the algebra of matrices. The few pages necessary for this purpose would be well spent, as some of the most important papers published at present on determinants are written in part in the language of matrices.*

One of the strong features of the book is the admirable selection of subjects; for though, as we have pointed out, there are a few things which one would like to have seen added, there is very little which one would be willing to see omitted. Open the book where you will and you find yourself in the midst of a live question having vital connections at the present day with other live branches of mathematics. This is no small praise when one recalls the possibilities for barren formalism which the subject of determinants presents. The choice not only of subjects but of the special aspects of these subjects to be considered and of the individual theorems to be proved has been guided by a true sense of values, not by the mere love of formal developments.

It is therefore somewhat surprising that in the selection of methods of proof a tendency to give the preference to formal manipulations is frequently apparent. Someone has said: "C'est le sentiment de se mouvoir dans l'identique qui fait la joie de l'algébriste" — a statement which would be correct if the word *formaliste* were put in place of *algébriste*. The algebraist has no less pleasure in taking large steps where the

* Cf., for instance, the paper by I. Schur already cited.

feeling of identity is lost than has the analyst or the geometer ; and indeed the criticism just made that many proofs in the book under review are of too purely formal a character is closely connected with the further fact that the author seems to make an effort to use as little algebra as possible. The conception of reducibility is nowhere used, and it is only in a brief note at the end of the volume (page 542) that the irreducibility of the general determinant is mentioned. Except in the case of the binary form, no reference is made to the fact that every polynomial can be resolved into irreducible factors in one and essentially only one way. Even such a fundamental fact as that a polynomial vanishes identically when and only when all of its coefficients are zero is not assumed to be known but is *proved* when, in the chapter on Elementary Divisors (page 258), it is found necessary.* Methods so thoroughly germane to the theory of determinants as are the elementary algebraic theorems just mentioned are avoided at the expense of simplicity and brevity. For instance, in the treatment of Vandermonde's determinant in Chapter IV, a formal proof covering two pages is given. This may perhaps be justified on the ground that at this early stage practice in applying the rules for manipulating determinants is desirable ; but a satisfactory reason can hardly be given for the omission of the brief and lucid proof which says that the determinant, being a polynomial in $a_1, \dots, a_n$, which vanishes when any two a 's are equal, must have each of the differences $a_i - a_j$ as a factor. The other factor, into which the product of these differences is multiplied, which is clearly a constant, can then be easily determined.†

Another good illustration of the tendency in question is to be found in the proof that the determinant of a skew symmetric matrix of even order is the square of a polynomial in the elements of the matrix. Two proofs of this theorem are given, the first of which covers a page and a half, and the second (which is of a purely formal character involving an examination of the individual terms of the determinant) three pages. The theorem may be proved by mathematical induction, a method also used in the first of the two proofs just mentioned, as follows :

* Contrast with this the free use made, without explanation, of uniform convergence, the integration of series term by term, and various matters relating to the differential and integral calculus.

† We must use here the algebraic theorem (or at least an easily proved special case of it) which says that if a first polynomial vanishes whenever a second irreducible one does, the second is a factor of the first.

Denote the determinant of order $2n$ by D , its cofactors by A , and the determinant obtained by striking out its last two rows and columns by S . Then

$$SD = A_{2n-1, 2n-1} A_{2n, 2n} - A_{2n, 2n-1} A_{2n-1, 2n}.$$

But since $A_{2n-1, 2n-1}$ and $A_{2n, 2n}$ are skew symmetric determinants of odd order and therefore vanish, while

$$A_{2n, 2n-1} = -A_{2n-1, 2n},$$

the above formula reduces to

$$SD = A_{2n, 2n-1}^2.$$

Consequently since S , being a skew symmetric determinant of order $2n - 2$, is by hypothesis the square of a polynomial in the elements, it follows, by an application of the theorem that a polynomial can be resolved into irreducible factors in essentially only one way, that the same is true of D .

The obvious reason for the second and longer proof is to introduce the subject of Pfaffians. This is, however, quite unnecessary. We may define the Pfaffian of a skew symmetric matrix of order $2n$ as the square root of its determinant, the sign being so determined that the Pfaffian reduces to $+1$ when

$$a_{12} = -a_{21} = 1, a_{34} = -a_{43} = 1, \dots, a_{2n-1, 2n} = -a_{2n, 2n-1} = 1$$

while all the other a 's are zero. This Pfaffian is then, by the theorem just proved, a polynomial of degree n in the a_v 's the forms and signs of whose terms can be determined with ease by methods which make far less demand on the concentrated attention of the reader than is called for in the section under consideration, and which also have the advantage of much greater brevity.

We have already indicated by various examples that the relations of different parts of the book to each other are not always clearly brought out. A minor, though not unimportant illustration of this same weakness is to be found in matters of terminology. The definition on page 78 of the reciprocal determinant does not tally with the definition on page 241 of the reciprocal quadratic form. The definition on page 49 of the rank of a finite determinant (or more accurately matrix) does not tally with the definition on page 399 of the rank of an

infinite determinant, nor with the definition on page 484 of the rank of a Fredholm determinant. Connected with a matrix of order n are two integers, its *rank* and its *nullity*.^{*} The sum of these two numbers is n , so that we may dispense with either one of them, and most authors use the rank alone. It is, however, sometimes convenient to use the nullity, and it is the generalization of this latter conception, not of the rank, with which we have to deal in the case of infinite determinants and of Fredholm determinants.

A more detailed table of contents containing section as well as chapter headings, and a more extensive index would have been of much assistance to users of the book.

Many will be inclined to criticise the scantiness of the bibliographical references placed, along with a few other notes, at the end of the volume. It must be remembered, however, that the collection of *reliable* information of this sort for such a widely ramified and, in the main, old field as that here treated is a very serious undertaking. An extensive and reliable bibliography would be, without doubt, a very welcome supplement, but it seems to be asking too much of the author of a book of this character to demand of him that he undertake such a task; the difficulties in producing good mathematical treatises are quite formidable enough without that.

We have said that the choice of subjects treated, both in the large and in detail, is admirable. The book has seemed to us at some points less satisfactory in the *methods* of proof selected, but these methods having once been chosen, the detailed exposition of the proof is always good and frequently a model of clearness and conciseness; it is evident that we have to deal with a writer whose expository powers are of the first order.

Last but not least, the standard of accuracy attained is remarkably high. In the sections which we have examined with care, and these constitute no inconsiderable portion of the book, the only error we have found is the one already mentioned of using the word *determinant* frequently in the sense of *matrix*, and this error is so sanctioned by universal usage that it must, for the present, be regarded rather as a defect of the age than of the individual. Only an author who has himself seriously tried to eliminate errors from his writings and is not satisfied, when a mistake is discovered, to say that he

^{*} The word *Defekt* has been used here in German.

wishes to be understood in some other sense, which his words do not admit (in other words to *crawl* if we may be permitted to use an expressive vulgarism) can appreciate the labor involved in the production of a text-book with this high record of correctness. We can recommend the book, in spite of some shortcomings, as one from which few readers can fail to get much valuable information without undue effort.

MAXIME BÔCHER.

DIFFERENTIAL INVARIANTS.

Invariants of Quadratic Differential Forms. By J. EDMUND WRIGHT. Cambridge Tract No. 9. Cambridge University Press, 1908. viii + 90 pp.

IN these days, when the number of papers in mathematics published each year is almost without limit and the ramifications are no less perplexing in their variety, one is delighted to find here and there a digest of the work in a particular field. These are the pleasures which the Cambridge Tracts hold in store for us, and we owe a debt of gratitude to our fellow-workers who are willing to pause in their researches to give us a panoramic view of their field and thus to turn over to us in nut-shell form the products of their searchings in the works of their collaborators. The author of the tract before us felt that this was its mission and he seems to have attained his hopes.

In his introduction he leads the reader into the domain of his subject by showing him an invariant and then defining it. The example taken is naturally enough the Gaussian curvature of a surface whose linear element

$$ds^2 = dx^2 + dy^2 + dz^2,$$

expressed in terms of two independent parameters u and v , is written

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2.$$

The idea of differential parameter is also suggested at this time. The reader is then acquainted with the magnitude of the field of inquiry and of the two kinds of problems: (i) The determination of all invariants of one or more differential quadratic forms and their relations; (ii) The geometrical and mechanical interpretation of these invariants.

In Chapter I the author gives a brief historical sketch of the development of the theory, indicating in particular the lines of attack and the leaders in these subfields. Here one finds almost all of the references to the memoirs in which the subject-matter is to be found. Subsequently the reader is inclined to feel that it would have been more helpful had the references been given along with the more or less extended development of the matter as set forth in the succeeding chapters. In these chapters, because of the limited amount of space, the treatment is necessarily brief and so it is desirable that the hurried reader should be able to dip at times into the original memoirs.

Chapter II opens with a short discussion of the problem of finding the necessary and sufficient conditions that two given surfaces be applicable to one another as an introduction to the general problem proposed by Christoffel * which may be stated as follows:

Given two differential quadratic forms in n variables

$$(1) \quad F = \sum_{i, \kappa} a_{i\kappa} dx_i dx_\kappa, \quad F' = \sum_{i, \kappa} a'_{i\kappa} dy_i dy_\kappa,$$

where $a_{i\kappa}$ and $a'_{i\kappa}$ are functions of the x 's and y 's respectively; what are the necessary and sufficient conditions that a transformation of the variables exist, say

$$(2) \quad x_i = \phi_i(y_1, \dots, y_n),$$

such that F is transformable into F' ?

This is equivalent to the determination of the conditions under which the system of partial differential equations

$$(3) \quad a'_{i\kappa} = \sum_{r, s} a_{rs} \frac{\partial x_r}{\partial y_i} \frac{\partial x_s}{\partial y_\kappa}$$

admits a solution. This question is reduced by Christoffel to an algebraic problem.

If equations (3) be differentiated with respect to y_l , where $l = 1, \dots, n$, and the resulting equations be solved for the second derivatives, the conditions that these equations be consistent are expressible in the form

$$(4) \quad (\alpha_1 \alpha_2 \alpha_3 \alpha_4)' = \sum_{i_1 i_2 i_3 i_4} (i_1 i_2 i_3 i_4) \frac{\partial x_{i_1}}{\partial y_{\alpha_1}} \frac{\partial x_{i_2}}{\partial y_{\alpha_2}} \frac{\partial x_{i_3}}{\partial y_{\alpha_3}} \frac{\partial x_{i_4}}{\partial y_{\alpha_4}},$$

* "Ueber die Transformation der homogenen Differentialausdrücke des zweiten Grades," *Crelle*, vol. 70 (1870).

where $(i_1 i_2 i_3 i_4)$ denotes a certain function of the quantities α_r and their first and second derivatives, and $(\alpha_1 \alpha_2 \alpha_3 \alpha_4)'$ a similar function in terms of the functions α'_r . These functions $(i_1 i_2 i_3 i_4)$ were used first by Riemann in the *Commentatio Mathematica* and so Ricci has called them the *Riemann symbols*.

Similarly to (3) equations (4) are the conditions of equivalence of a form

$$(5) \quad G_4 = \sum_{i_1 i_2 i_3 i_4} (i_1 i_2 i_3 i_4) d^{(1)} x_{i_1} d^{(2)} x_{i_2} d^{(3)} x_{i_3} d^{(4)} x_{i_4},$$

and an analogous form G'_4 in the y 's. Here we take four different sets of differentials, since there are certain linear relations between Riemann symbols, in consequence of which G_4 vanishes identically when the differentials are the same. Thus equations (4) are the conditions for the equivalence of two *quadrilinear* forms.

If equations (4) be differentiated with respect to y_a and if we make use of five index symbols defined by

$$(6) \quad \begin{aligned} (i_1 i_2 i_3 i_4 i) &= \frac{\partial}{\partial x_i} (i_1 i_2 i_3 i_4) \\ &- \sum_{\lambda} \left[\left\{ \begin{matrix} i i_1 \\ \lambda \end{matrix} \right\} (\lambda i_2 i_3 i_4) + \left\{ \begin{matrix} i i_2 \\ \lambda \end{matrix} \right\} (i_1 \lambda i_3 i_4) + \dots \right], \end{aligned}$$

where $\left\{ \begin{matrix} i j \\ \lambda \end{matrix} \right\}$ is the usual Christoffel symbol, we obtain equations

$$(7) \quad (\alpha_1 \alpha_2 \alpha_3 \alpha_4)' = \sum_{i_1 i_2 i_3 i_4 i} (i_1 i_2 i_3 i_4 i) \frac{\partial x_{i_1}}{\partial y_{a_1}} \dots \frac{\partial x_i}{\partial y_a},$$

which evidently are the conditions for the equivalence of two *quinquilinear* forms G_5 and G'_5 .

Continuing by means of a process which is an immediate generalization of (6), we obtain a series of equations analogous to (7) and thus a sequence of covariant forms $G_4, G_5, \dots, G_\mu$.

The author gives few details in connection with the statement of the foregoing results, but he develops at some length the proof of the following fundamental theorem of Christoffel:

The necessary and sufficient conditions in order that it shall be possible to transform a quadratic form F into another quadratic form F' are that the equations in the variables $x, y, \partial x / \partial y$ derived from the equivalence of two sequences

$$F, G_4, G_5, \dots, G_\mu \quad \text{and} \quad F', G'_4, G'_5, \dots, G'_\mu$$

shall be algebraically compatible.

Wright shows that two cases are to be considered :

(i) When an order q can be determined such that the equations

$$F' = F'', \quad G_1 = G'_1, \quad \dots, \quad G_{q-1} = G'_{q-1}$$

determine the quantities x and $\partial x / \partial y$ as functions of the y 's and these values make G_q and G'_q an identity.

(ii) When the above equations yield only $p (< n^2 + n)$ independent equations, and if any set of solutions of these p equations satisfy identically $G_q = G'_q$.

When the conditions of the first case are satisfied the problem is algebraic, but in the second case certain of the quantities left undetermined by the algebraic conditions must satisfy partial differential equations of the first order. From the algebraic theory it follows that the algebraic invariants of the two sets of multilinear forms must be equal. And the equations of transformation are given by equating n independent absolute invariants. Hence it is necessary that the sequence of forms $F, G_1, G_2, \dots$ be extended to such a point that there shall be n absolute simultaneous invariants.

In particular the invariants $I, I_1, \dots$ of the algebraic forms $F, G_1, \dots$ are a complete system of relative differential invariants of the form F' , and if under any transformation I becomes κI , then κ is some power of the Jacobian of the transformation.

The remainder of Chapter II is devoted to a discussion of the "absolute differential calculus" which Ricci and Levi-Civita* have developed from the fundamental principles discovered by Christoffel.

Equations (3) may be looked upon as defining a transformation of the functions a_{rs} simultaneously with the transformation (2) of the variables. The same may be said of equations (4) and (7). All of these equations are of the form

$$(8) \quad Y_{r_1 \dots r_m} = \sum_{s_1 \dots s_m} X_{s_1 \dots s_m} \frac{\partial x_{s_1}}{\partial y_{r_1}} \dots \frac{\partial x_{s_m}}{\partial y_{r_m}}.$$

Functions $X_{s_1 \dots s_m}$ whose transformation equations are of this sort are said to form a *covariant system* of order m . In similar manner, the functions $X^{(s_1 \dots s_m)}$ whose transformation equations

* *Math. Annalen*, vol. 54, pp. 125-201.

are of the form

$$Y^{(r_1 \dots r_m)} = \sum_{s_1 \dots s_m} X^{(s_1 \dots s_m)} \frac{\partial y_{r_1}}{\partial x_{s_1}} \dots \frac{\partial y_{r_m}}{\partial x_{s_m}}$$

form a *contravariant system* of order m . As an example, the coefficients $\alpha^{(r)}$ of the quadratic form reciprocal to F form a contravariant system of order two.

Equations (8) are the conditions that the multilinear forms

$$\sum_{r_1 \dots r_m} Y_{r_1 \dots r_m} dy_{r_1} \dots dy_{r_m}, \quad \sum_{s_1 \dots s_m} X_{s_1 \dots s_m} dx_{s_1} \dots dx_{s_m}$$

shall be transformable into one another. They are called *associated forms*.

If $X_{r_1 \dots r_m}$ and $\Xi_{r_1 \dots r_m}$ are typical terms of two covariant systems, then $X_{r_1 \dots r_m} + \Xi_{r_1 \dots r_m}$ is a term of a covariant system of the same order; in similar manner the quantities $X_{r_1 \dots r_m} \Xi_{s_1 \dots s_p}$ form a covariant system of order $m + p$. Analogous results are true of contravariant systems.

Of particular importance is the idea of *composition* of a covariant and a contravariant system. Thus, the two systems whose typical functions are

$$\sum_{s_1 \dots s_p} \Xi^{(s_1 \dots s_p)} X_{r_1 \dots r_m s_1 \dots s_p}, \quad \sum_{s_1 \dots s_p} \Xi_{s_1 \dots s_p} X^{(r_1 \dots r_m s_1 \dots s_p)}$$

can be shown to be covariant and contravariant respectively of order m .

The particular system $X^{(r_1 \dots r_m)}$ defined by

$$\sum_{s_1 \dots s_m} \alpha^{(r_1 s_1)} \dots \alpha^{(r_m s_m)} X_{s_1 \dots s_m}$$

is said to be the reciprocal system of $X_{r_1 \dots r_m}$ with respect to the fundamental form F ; and conversely.

There is one other fundamental idea namely *covariant and contravariant differentiation* with respect to a form F . The former is a generalization of equation (6) and the latter is analogous to it. It is significant that if $X_{r_1 \dots r_m}$ be a term of a covariant system, the functions $X_{r_1 \dots r_m \lambda}$ and $\tilde{X}_{r_1 \dots r_m \lambda}$, obtained from the first by repeated covariant differentiation with respect to x_λ and $\tilde{x}_\lambda$, and inversely, are not equal in general.

Ricci and Levi-Civita have established by these means the following fundamental theorem :

In order to obtain all the absolute differential invariants of order μ involving the coefficients of F and functions $f_1, \dots, f_r$, it suffices to determine the algebraic invariants of the following system of forms

$$(i) \quad F, G, \dots, G_{\mu-2};$$

(ii) the associated forms obtained from $f_1, \dots, f_r$ with respect to F up to order μ .

Wright merely states the general theorem and illustrates the method of proof in the case of a binary form.

In 1884 Lie applied his theory of continuous groups to the determination of the differential invariants of a binary quadratic differential form. This necessitated the introduction of the idea of an "extended" group, consisting of the transformations operating not only upon the original variables x and y , but also upon such added variables, as the coefficients E, F, G of the quadratic form $Edx^2 + 2Fdx dy + Gdy^2$ and certain functions $f_1, \dots, f_r$ and their derivatives with respect to x and y . This requires a knowledge of the variation of these quantities due to an infinitesimal transformation of the original group, written

$$x' = x + \xi(x, y)\delta t, \quad y' = y + \eta(x, y)\delta t,$$

where ξ and η are arbitrary. Then if Ω is the infinitesimal operator of the group in the variables $x, y, E, F, G, f, f_x, f_y$,

$$\Omega \equiv \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \frac{\delta E}{\delta t} \frac{\partial}{\partial E} + \dots + \frac{\delta f}{\delta t} \frac{\partial}{\partial f} + \dots + \frac{\delta f_y}{\delta t} \frac{\partial}{\partial f_y},$$

and an invariant I must satisfy the condition $\Omega I = 0$, whatever be ξ and η . Wright calculates the quantities $\delta E, \dots, \delta f_y$ directly and gives the complete system of linear equations in the above eight variables which I must satisfy. This is followed by an explanation of the case where the invariants involve functions f and their derivatives of order p , and an exposition of the general method of Forsyth for the calculation of the increments of a function.

The remainder of Chapter III deals with the determination of differential parameters for forms of rank zero, that is forms for which we can find functions u_λ satisfying the condition

$$(9) \quad \sum_{r,s}^* a_{rs} dx_r dx_s = \sum_{\lambda}^* du_{\lambda}^2.$$

The variables entering are

- (i) $u_1, u_2, \dots, u_n$ and their total differentials;
- (ii) n , or less, arbitrary functions $\phi^{(i)}$ and their derivatives;
- (iii) the Jacobian J of the u 's.

If we let F_m denote

$$\left\{ \sum_i^n U_i \frac{\partial}{\partial u_i} \right\}^m \phi,$$

where the U 's are a set of auxiliary variables, the function F_m is an algebraic form of order m whose coefficients are the m th derivatives of ϕ except for numerical multipliers. It is shown that the functions F are expressible as functions of the quantities α_m and their derivatives. The final result is:

The most general invariant is a function of the quantities $d\phi^{(i)}$, the general algebraic invariants of the forms F and of ΣU^2 , multiplied by some power of J .

In general more than n functions u_λ are necessary in order that the F may be given the form (9). If there are m ($> n$), then F may be looked upon as a manifold of order n in the space of order m . In this case the problem of finding the invariants of F is that of determining which of the invariants of

$$\sum_1^m du_\lambda^2$$

are invariants of the manifold.

In several memoirs Maschke has developed a symbolic method for differential forms analogous to the Clebsch method for algebraic forms. In Chapter IV Wright gives an exposition of this method. The assumption is made that the quadratic form (1) can be expressed symbolically as $(df)^2$, where f is a symbolic function of the n variables. The derivatives of f , written $\partial f / \partial x_i \equiv f_i$, have no meaning separately, but $f_i f_j$ is to mean α_{ij} . As in the case of algebraic forms, equivalent symbols must be introduced in number to the order of the α 's appearing in an expression. If $\Phi^{(i)}$, for $i=1, \dots, n$, are n invariant expressions of the quadratic form F , so that $\Phi^{(i)} = \Phi'^{(i)}$, where $\Phi'^{(i)}$ is $\Phi^{(i)}$ for the new set of variables y , it is shown that the function

$$(10) \quad |a_{11} a_{22} \dots a_{nn}|^{-\frac{1}{2}} \left| \frac{\partial \Phi^1}{\partial x_1} \frac{\partial \Phi^2}{\partial x_2} \dots \frac{\partial \Phi^n}{\partial x_n} \right|$$

is an absolute invariant. This function, denoted by (Φ) , is called an *invariantive constituent of the form F* . Any combination of these constituents which has a significance in terms of the α 's is an invariant, and thus the invariants are formed.

By (f) is meant the invariantive constituent of equivalent symbols expressing the quadratic form; by (uf) the constituent in which u is any function of the n variables and the remainder of the Φ 's in (Φ) are equivalent symbols. In particular, we have

$$(f)^2 = n!, (uf)^2 = (n-1)! \Delta_1 u, (uf)(vf) = (n-1)! \Delta(u, v), \\ ((uf)f) = (n-1)! \Delta_2 u,$$

where $\Delta_1 u$, $\Delta(u, v)$ and $\Delta_2 u$ are the well-known differential parameters of F . In deriving the last of these identities use is made of covariant differentiation which is expressible thus

$$u^{(\kappa)} = u_{\kappa} - \frac{1}{(n-1)!} f_{\kappa}(fa)(ua),$$

f and a being equivalent symbols for the fundamental form, and u_{κ} denoting

$$\frac{\partial^2 u}{\partial x_i \partial x_{\kappa}}.$$

In the reduction of the invariants to suitable form Maschke makes use of a large number of symbolic identities a few of which Wright gives. The Riemann symbols are expressible in an interesting form, namely

$$(i\kappa rs) = f_{ir} f^{(\kappa s)} - f_{is} f^{(\kappa r)},$$

the subscripts referring to ordinary differentiation of the symbolic function f and the superscripts to covariant differentiation. By means of these symbolic expressions for the covariant forms G_4 and G_5 are obtained. This chapter is very short, and is in fact a resumé with few details of Maschke's memoirs in the fourth volume of the *Transactions*.

The final chapter of the book, which constitutes about half of it, is devoted to applications. The first twelve pages deal with differential parameters of a single form for $n = 2$, the subject matter being similar to that which may be found in Volume III of Darboux's *Leçons*. The remainder sets forth some of

the applications of the absolute differential calculus. The functions used belong to covariant or contravariant systems, and the ordinary operation of differentiation is replaced by covariant or contravariant differentiation with respect to a fundamental form.

The author contributes a solution by these means of the problem of determining the most general triply orthogonal system of surfaces in a three dimensional manifold with a given quadratic form, and arrives at a differential equation of the third order which is an interesting generalization of the equation found by Darboux for euclidean space. The other applications are chosen from those given by Ricci and Levi-Civita. We shall call attention to several of them.

If $\lambda^{(1)}, \dots, \lambda^{(n)}$ be n functions of $x_1, \dots, x_n$ satisfying the condition

$$(11) \quad \sum_{ik} a_{ik} \lambda^{(i)} \lambda^{(k)} = 1,$$

the n equations

$$\frac{dx_i}{ds} = \lambda^{(i)}$$

define a family of curves, one through each point, which consequently may be said to be a congruence. It is readily seen that the coefficients $\lambda^{(i)}$ form a contravariant system. If λ_i denotes the reciprocal function of $\lambda^{(i)}$ then equation (11) may be written in the other forms

$$\sum_{i,k} a^{(ik)} \lambda_i \lambda_k = 1, \quad \sum_r \lambda^{(r)} \lambda_r = 1.$$

If we have n congruences such that the functions $\lambda_{h/r}$ and $\lambda_{k/r}$ (where h and k signalize the congruence and r the derivative) satisfy the conditions

$$\sum_r \lambda_h^{(r)} \lambda_{k/r} = \eta_{hk} \quad (h, k = 1, \dots, n),$$

where $\eta_{hk} = 0$ when $h \neq k$, and $\eta_{hh} = 1$, the set of congruences is said to form an *orthogonal ennuple*. An important theorem is that any covariant or contravariant system is expressible in terms of invariants and the coefficients of an orthogonal ennuple. In particular, the first covariant derivatives of the λ 's may be expressed in terms of themselves and certain invariants γ_{hki} , thus

$$\lambda_{h/r} = \sum_{k,i} \gamma_{hki} \lambda_{k/r} \lambda_{i/r}.$$

These invariants satisfy the conditions

$$\gamma_{\lambda\lambda q} = 0, \quad \gamma_{\lambda k q} + \gamma_{k\lambda q} = 0,$$

and so there are $\frac{1}{2}n^2(n-1)$ of them. They are a generalization of the rotations p, q, r, p_1, q_1, r_1 of the moving axes of Darboux, and like the latter they must satisfy certain partial differential equations of the first order.

The necessary and sufficient condition that a congruence $\lambda_{n/s}$ admit of a family of orthogonal surfaces is

$$\gamma_{nkh} = \gamma_{nhk} \quad (k, h = 1, \dots, n-1).$$

A geodesic being defined as a curve for which the first variation of the integral of F in (1) is zero, the condition necessary and sufficient that a congruence $\lambda_{n/s}$ consist of geodesics is that $\gamma_{nri} = 0$, for $i = 1, \dots, n$.

The absolute calculus may be applied also to dynamical problems. Consider any holonomic system and let its generalized coordinates be $x_1, \dots, x_n$. The kinetic energy of the system, say T , is given by

$$2T = \sum_{r,s} a_{rs} \dot{x}_r \dot{x}_s,$$

where the dot denotes differentiation with respect to the time. If the system is subject to external forces depending only on the position, so that the work done in a small displacement is given by

$$\sum_r X_r dx_r,$$

the equations of motion are those of Lagrange

$$(12) \quad \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_r} \right) - \frac{\partial T}{\partial x_r} = X_r.$$

The given dynamical problem may be regarded as solved when we know the most general values of the x 's as functions of t which satisfy (12). These equations define a curve in n -space which is called a *trajectory* of the configuration. By choosing a congruence of these trajectories as one of the systems of an orthogonal ennuple the equations of Lagrange may be given an invariant form of great simplicity. In particular, it may be shown that if the external forces are all zero, the tra-

jectories are the geodesics of the manifold, and they are described with constant velocity. An important result is that the knowledge of an algebraic integral of the system (12) carries with it that of a homogeneous integral of the differential equations for geodesics in the manifold. The chapter closes with a discussion of homogeneous linear and quadratic integrals of the equations of geodesics; the determination in invariant form of the criteria that two dynamical systems have the same trajectories; and the geodesic representation of one manifold upon another.

The subject matter is presented in an inspiring way, so that it seems very probable that the reader will turn to the papers of Ricci and Levi-Civita, as the author hopes. The proof reading has been well done and in every way the printer's work is satisfactory.

It is impossible to close this review without remarking the loss to American mathematics by the death of Mr. Wright. His brilliant record at Cambridge and his subsequent career in this country had won for him a high place in his field.

LUTHER PFAHLER EISENHART.

STURM'S GEOMETRISCHE VERWANDTSCHAFTEN.

Die Lehre von den geometrischen Verwandtschaften. Vierter Band: die nichtlinearen und die mehrdeutigen Verwandtschaften zweiter und dritter Stufe. By RUDOLF STURM. Leipzig and Berlin, Teubner, 1909. x + 486 pp.

As the subject matter of this fourth volume of Professor Sturm's extensive treatise on geometric relations is so different from that of the preceding ones,* but little analogy can be drawn with the methods already discussed. With the exception of one elementary treatise, mathematical literature did not include a book on birational transformations before the appearance of the present volume, although over five hundred memoirs have been devoted to the subject during the last two decades. The newness of the subject, the possibility of approaching it from different standpoints, and its applicability to so many other

* These volumes have been reviewed in the BULLETIN; volume 1 in vol. 15, p. 135; volume 2 in vol. 15, p. 252; volume 3 in vol. 16, p. 250.

disciplines all contribute to make the task of preparing a systematic treatise both important and difficult. Some aspects must be omitted altogether and others but briefly sketched. As to the wisdom of any particular choice of material there will doubtless be a wide diversity of opinion, for methods that are of immediate use for one purpose are of no value for another, although both serve to develop the same theory. The present volume proceeds synthetically, with a slight digression into more algebraic methods in discussing the general linear transformations of a complex variable. In most cases a brief outline only is given, the author constantly referring the reader to the original memoirs for the details; over three hundred such references are supplied. As in the preceding volumes, a glossary of technical terms is provided, the present one containing 74 terms.

In the earlier volumes a number of illustrations of Cremona transformations were met; in some cases the image of a straight line was actually found, and the general theorem that the genus of a curve is invariant under any birational transformation of the plane was proved (volume 1, no. 163). The present volume begins with the problem of finding the most general net of curves that can be imaged on the ∞^2 straight lines of the plane. Several theorems regarding the number of constants which determine a curve, the intersections of curves, and the law of equivalence of multiple points are stated as lemmas; from these criteria and the fact that the curves are rational it follows that each curve must be fixed by two linear conditions and that any two curves intersect in one point apart from the basis points of the net. The discussion occupies 20 pages, contains all the fundamental relations, includes a complete table of forms of nets up to order 10, and is admirably well presented. The next ten pages are devoted to illustrations, containing a veritable mine of suggestions for further research. The chapter on quadratic inversion is unnecessarily long (33 pages) and detailed; the various forms of the principal elements are not kept sharply separate. Conformal representation and transformation by reciprocal radii receive a great amount of attention. At the end of this chapter the complex variable is introduced and the preceding theory is developed analytically. The involutorial (simple) inversions are consistently distinguished from general non-periodic ones, but it is not shown that the latter can be obtained from the former by multiplication with a collineation.

Involutorial transformations of higher order are now discussed, largely following Doehlemann, but the Geiser transformation, fixed by a net of general cubic curves having seven distinct basis points, is also treated.

The first chapter closes with the theory of correspondence applied to ternary fields. The general formula for the number of coincidences is derived, then confirmed by numerous illustrations, taken from cases in which the number of coincidences is determined in another way.

The second chapter is devoted to correspondences on curves of genus one. Most of it is confined to the general plane cubic, then the space quartic of the first kind, the plane binodal quartic and the ruled quartic surface having two double directrix lines are considered. Each of these entities can be rationally mapped on any one of the others. The treatment is essentially that of Weyr, and is very complete, containing a full geometric theory of the properties of cubic curves and a well drawn picture of the configurations which define finite Cremona cycles, as well as of the G_{18} of collineations under which a non-singular, cubic curve remains invariant.

The problem of multiple correspondences between ternary fields is again taken up in an instructive chapter which considers in detail the $(1, 2)$ case. The procedure is analogous to that of Paolis, but the latter is only an abstract of the wealth of material and particularly of new view points here developed. If the genus of the curves in the simple field which are the images of the straight lines in the double field is called the genus of the transformation, it is shown that we need only consider the two cases, genus zero and genus one. The basis points, the coincidence curve of the involution in the simple field, and the contact curves of the curve of branch points in the double field are exhaustively treated. A typical case is the Geiser transformation, already discussed in the preceding chapter. A later section considers $(m, 1)$ and $(2, 2)$ correspondences. Finally, a number of illustrations are added to show the usefulness and power of the ideas here developed. An immense field of uncultivated territory is opened by this discussion.

As a suitable transition from Cremona transformations in the plane to those in space, a chapter is introduced which treats of those surfaces which can be rationally mapped upon the plane. The general stereographic projection of a quadric surface is

followed by a much fuller discussion of the case of the cubic. Two general methods are considered, that of trilinear pencils of planes and that of the linear congruence whose directrices are two skew lines lying on the surface. This is followed by the Steiner surface, the cubic ruled surfaces, the quartic having a double conic and the quintic having a double space cubic. Finally, the depiction of a surface of order n which has an $n-2$ -fold line and the Noether surface of order 4 are discussed. One interesting and important question in this connection is hardly touched upon, namely, to determine the nature of those transformations in space which go into Cremona transformations in the plane under which a plane curve remains invariant.

The birational transformations of space are introduced by making a system of surfaces having four homogeneous linear parameters collinear with the planes of space. The images of lines in one space and of planes in the other are shown to be of the same order, but those of planes in the first and of lines in the second are of an order independent of the first number, being fixed by the configuration of the principal elements. In order that the operations be reversible, the principal elements must absorb all but one of the intersection of any three surfaces of the system. The principal points, curves, and surfaces are brought in, together with the principal curves of the second kind, i. e., the locus of points whose images are all the same principal curve. The entire configuration is shown to constitute the Jacobian of the system. The images of lines in either system must be rational curves, and of planes must be homaloidal surfaces.

This short introduction is then amplified by studying various particular cases in detail, the first being a system of quadrics passing through four fixed points and touching a fixed plane at one of them. The system in the other space is composed of Steiner surfaces. When the quadrics have a line in common, the Steiner surfaces are replaced by cubic ruled surfaces. If the quadrics have a conic in common, the other system is also composed of quadrics. A particular form of this case is the transformation by reciprocal radii, which is discussed at some length. The second illustration is that of general cubic surfaces. It begins with Meusnier's theorem regarding the curvature of plane normal sections of a surface, in order to provide for an intersection of two cubics of the desired form. On account of the number of forms of the cubic surface this case

receives a long discussion. An extensive knowledge of the forms of cubic surfaces is presupposed, the author constantly referring to his own book on this subject.

The chapter on involutorial transformations is an enumeration of a number of the better known cases, such as generalized quadratic inversion, the polarity of a cubic surface from a point upon it, the harmonic conjugate of a point with regard to the points of intersection of a line through it and cutting a given cubic curve twice, and the pairs of associated points defining a bundle in a system of ∞^3 quadric surfaces passing through six fixed points, this last case being the Geiser transformation of space. The peculiarities of the fifteen lines joining the six points by twos and the cubic curve passing through all of them are discussed in detail. These lines are interesting as furnishing the first example of a principal curve of the second kind. The more general investigations of Montesano are not considered in the present volume.

The last chapter treats of multiple correspondences in space of three dimensions. It begins with the determination of the number of coincidences, and confirms the result by numerous simple illustrations. A short discussion is devoted to correspondences in line space; the general formula for the number of coincidences is derived and a simple illustration given. As in the case of the plane, the next section considers more in detail the (2, 1) correspondence first studied by Paolis. Most of the results are directly analogous in space to those obtained above for the plane, but to follow the proofs a knowledge of the author's treatise on line geometry is necessary. This interesting section is followed by two cases of (2, 2) quaternary correspondences, and a generalization of the duality defined by a linear complex (höhere Nullverwandtschaft). A short appendix completes the proofs of a few theorems in the preceding volumes and extends a few results to more general cases.

VIRGIL SNYDER.

NOTES.

THE Annual Register of the AMERICAN MATHEMATICAL SOCIETY is now in preparation and will be issued in January. Blanks for furnishing necessary information have been sent to the members. Early notice of any changes since the issue of the last Register will greatly facilitate the work of the Secretary. The

Register is widely circulated and it is desirable that the information which it contains should be accurate and reliable.

THE concluding (October) number of volume 11 of the *Transactions of the American Mathematical Society* contains the following papers: "Conjugate line congruences contained in a bundle of quadric surfaces," by V. SNYDER; "On the fundamental number of the algebraic number-field $k(\sqrt[m]{m})$," by J. WESTLUND; "Volterra's integral equation of the second kind, with discontinuous kernel," by G. C. EVANS; "Ein Seitenstück zur Möbius'schen Geometrie der Kreisverwandtschaften," by H. BECK; "Vector interpretation of symbolic differential parameters," by L. INGOLD; "Surfaces with isothermal representation of their lines of curvature and the transformations (second memoir)," by L. P. EISENHART; "On the base of a relative number-field, with an application to the composition of fields," by G. E. WAHLIN; "The strain of a non-gravitating sphere of variable density," by L. M. HOSKINS.

THE concluding (October) number of volume 32 of the *American Journal of Mathematics* contains the following papers: " q -difference equations," by F. H. JACKSON; "On the relation between the sum-formulas of Hölder and Cesàro," by W. B. FORD; "Sur un exemple de fonction analytique partout continue," by D. POMPIEU; "Symmetric binary forms and involutions," by A. B. COBLE; "Systems of tautochrones in a general field of force," by A. W. REDDICK; "The general transformation theory of differential elements," by E. KASNER.

THE opening (October) number of volume 12 of the *Annals of Mathematics* contains the following papers: "The straight line solutions of the problem n bodies," by F. R. MOULTON; "On semi-analytic functions of two variables," by MAXIME BÔCHER; "Some theorems concerning systems of linear partial differential expressions," by W. J. BERRY; "Some circles associated with coneylic points," by J. L. COOLIDGE; "On a method for the summation of series," by R. E. GLEASON.

AT the annual meeting of the Cambridge philosophical society on October 31 Mr. G. H. HARDY read a paper on "Fourier's double integral and the theory of divergent integrals."

THROUGH the gift of Mr. John Claffin the College of the City of New York has acquired the library of the late SIMON NEWCOMB, consisting of 4,000 books and 6,000 pamphlets.

ON September 4 more than a hundred participants founded

the Swiss mathematical society as a section of the Society of Swiss naturalists. The affairs of the society will be conducted by a council of three members, elected for a term of two years. The present council consists of Professor R. FUETER, of Basel, chairman, Professor H. FEHR, of Geneva, and Professor M. GROSSMAN, of Zürich, secretary. The first session was held September 6, at which the following papers were read: M. GROSSMAN, Zürich: "A geometric problem of photogrammetry." R. FUETER, Basel: "Principles of classification of algebraic numbers." F. PRASEL, Zürich: "Graphical methods in hydrotechnical problems." O. SPIESS, Basel: "Geometric considerations." D. MIRIMANOFF, Geneva: "The last theorem of Fermat." H. FEHR, Geneva: "Report of the international commission." E. MEISSNER, Zürich: "Concerning a surface not in tetrahedral position." F. RUDIO, Zürich: "Report on the publication of the works of Euler." R. LAEMMEL, Zürich: "Mathematics and biology."

THE firm of B. G. Teubner in Leipzig will celebrate the completion of its first century on March 3, 1911. A memorial address for the occasion is being prepared by Dr. F. SCHULZE; it will contain a biography of the founder of the firm, an account of its early activity in literature, and a detailed presentation of the scientific development of the publishing house since 1850.

AN excellent heliographic reproduction has been made of the photograph of Dirichlet mentioned in the BULLETIN, volume 15, page 318. Copies can be had for 3.35 marks by applying directly to Frä. Lotte Nelson, Darmstadt, Landskronstrasse 39.

UNIVERSITY OF PARIS. The following mathematical courses are announced for the semester beginning November 3, 1910:—By Professor G. DARBOUX: Infinitesimal geometry, cartography, two hours.—By Professor E. GOURSAT: Differential and integral calculus, elements of the theory of analytic functions, two hours.—By Professor E. BOREL: Integral functions, one hour.—By Professor P. APPELL and M. C. GUICHARD: General laws of equilibrium and motion, two hours.—By MM. CARTAN and BLUTEL: General mathematics, first part, two hours.—By Professor H. POINCARÉ: Cosmogonic hypotheses, two hours.—By Professor J. BOUSSINESQ: Mechanical theory of light, two hours.—By Professor G. KOENIGS: General theory of mechanisms, two hours.

Conferences will be conducted by Professor KOENIGS and MM. GUICHARD, CARTAN, BLUTEL, SERVANT, and ROUBAUDI.

In the Ecole Normale. . By Professor J. TANNERY : Differential and integral calculus.—By Professor E. BOREL : Theory of functions.

MR. J. M. DODDS of Peterhouse, Mr. J. B. PEACE of Emmanuel College, and Mr. P. C. GAUL of Trinity College have been appointed examiners for Part I of the mathematical tripos at Cambridge for 1911.

ON the invitation of the committee of the Wolfskehl foundation, Professor H. A. LORENTZ, of the University of Leyden, delivered a course of lectures on the development of our ideas of the ether at the University of Göttingen during the week of October 24–29.

DR. E. FANTA has been appointed docent in mathematics and insurance at the German technical school at Brünn.

DR. H. MOHRMANN has been appointed docent in mathematics at the technical school of Karlsruhe.

DR. ROTHE has been appointed docent in mathematics at the technical school of Vienna.

DR. V. N. ROSEVARE has accepted the professorship of mathematics in the University of South Africa at Natal.

MR. H. BATEMAN, fellow of Trinity College, Cambridge, and reader in mathematical physics at the University of Manchester, has been appointed lecturer in mathematics at Bryn Mawr College.

AT Ohio Wesleyan University Professor G. N. ARMSTRONG has been promoted to a full professorship of mathematics.

DR. H. A. CONVERSE has been appointed head of the department of mathematics in the Baltimore Polytechnic Institute.

AT Harvard University Professor E. B. WILSON, of the Massachusetts Institute of Technology, has been appointed lecturer in mathematics for the second half of the present academic year. He will give a course on the "Applications of probabilities to theoretical physics." Dr. F. J. DOHMEN has been appointed instructor in mathematics.

AT the University of Colorado Professor S. EPSTEEN has been promoted to a full professorship of mathematics.

PROFESSOR R. B. MCCLENON, of Iowa College, has been promoted to an associate professorship of mathematics.

AT Bryn Mawr College Dr. ISABEL MADDISON has been appointed to the office of recording dean.

AT the College of the City of New York Dr. F. M. PEDERSEN has been promoted to an assistant professorship of mathematics.

DR. ALEXANDER PELL of the Armour Institute has been promoted from an assistant professorship to an associate professorship of mathematics.

DR. H. W. STAGER, of the University of California, has been appointed instructor in mathematics at Fresno Junior College.

DR. E. W. SHELDON, of Yale University, has been appointed assistant professor of mathematics in the University of Alberta.

MR. MEYER GABA, of the University of Kansas, has been appointed instructor in mathematics in the school of mines of the University of Missouri, Rolla, Mo.

DR. H. F. MACNEISH, of Princeton University, has been appointed instructor in mathematics in the Sheffield Scientific School of Yale University.

MR. H. PRITCHARD and Mr. J. PITMAN have been appointed assistants in mathematics at Swarthmore College.

MR. R. DUHADWAY has been appointed instructor in mathematics at the University of Iowa. Mr. F. EATON has resigned his instructorship to accept a position in the new government college at Peking.

MR. R. L. CARY has been appointed instructor in mathematics at Princeton University.

AT the University of Georgia Dr. R. P. STEPHENS has been promoted from an adjunct professorship to an associate professorship of mathematics. Professor J. F. MESSICK, late of Randolph Macon College, has been appointed adjunct professor of mathematics.

DR. J. K. LAMOND, of Yale University, has been appointed instructor in mathematics at Wesleyan University.

AT Brown University Dr. R. C. ARCHIBALD has resumed his academic work after a year's absence in Paris. Mr. F. W. BEAL has been appointed instructor in mathematics. Mr. C. H. CURRIER has a year's leave of absence for study at Göttingen.

MISS IDA M. SCHOTTENFELS has been appointed head of the department of mathematics in Toledo University.

DR. A. E. HAYNES, professor of engineering mathematics in the University of Minnesota, has been granted a year's leave of absence. Professor Haynes has taught consecutively for thirty-five years, fifteen years at Hillsdale College, three years in the Michigan College of Mines, and seventeen years in the University of Minnesota.

PROFESSOR G. A. OSBORNE, of the Massachusetts Institute of Technology has been made professor emeritus.

PROFESSOR A. B. NELSON, of Central University, has retired from active service.

PROFESSOR J. LÜROTH, of the University of Freiburg, died September 14, at the age of 66 years.

PROFESSOR W. THOMÉ, of the University of Greifswald, died October 1, at the age of 69 years.

PROFESSOR P. WEINMEISTER, of the academy of forestry at Tharandt, died August 27, at the age of 62 years.

NEW PUBLICATIONS.

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- ARENS (W.). Latein oder Deutsch? Die Sprachenfrage bei der Herausgabe der Werke Leonhard Eulers. Magdeburg, Peters, 1910. 8vo. 4 + 76 pp. M. 1.60
- BACHMANN (P.). Zahlentheorie. Versuch einer Gesamtdarstellung dieser Wissenschaft in ihren Hauptteilen. In 6 Teilen. Teil I: Die Elemente der Zahlentheorie (1892). Anastatischer Neudruck. Leipzig, Teubner, 1910. M. 7.20
- BARBETTE (E.). Les sommes de p^{m+1} puissances distinctes égales à une p^{m+1} puissance. Avec une table des 5000 premiers nombres triangulaires. Liège, Gnuse, 1910. 4to. 164 pp. Fr. 12.50
- BIANCHI (L.). Vorlesungen über Differentialgeometrie. Uebersetzt von M. Lukat. 2te vermehrte und verbesserte Auflage. Leipzig, Teubner, 1910. 8vo. 18 + 721 pp. Cloth. M. 24.60
- BOGRAD (J.). Die Astroidenfläche. (Diss.) Bern, Suter, 1909. 4to. 40 pp. Fr. 1.00
- BURALI-FORTI (C.) ET MARCOLONGO (R.). Éléments de calcul vectoriel avec de nombreuses applications à la géométrie, à la mécanique et à la physique mathématique, traduit de l'italien par S. Lattès. Paris, Hermann, 1910. 8vo. 229 pp. Fr. 8.00
- BURGGRAF (G.). Die 1'-Funktion für komplexe Argumente. Schluss. (Progr.) Brünn, 1910. 8vo. 29 pp.
- CARNOY (H.). Sur les travaux mathématiques de M. Ernest Lebon (Grands dictionnaires biographiques internationaux). Paris. Pouget, 1910. 16mo. 65 pp.

- DINTZI (E.). Ueber einige Eigenschaften der Bernoullischen und analoger Zahlen. (Progr.) Wien, 1910. 8vo. 11 pp.
- EGLI (M.). Beitrag zur Theorie unendlicher Systeme linearer Gleichungen mit unendlich vielen unbekanntem. (Diss.) Zürich, Lee-
mann, 1910. 8vo. 61 pp.
- ENCYCLOPÉDIE des sciences mathématiques pures et appliquées. Edition française dirigée par J. Molk. Tome I, volume 3: Théorie des nombres. Fascicule 3: Théorie arithmétique des formes, par Vahlen et Cahen. Propositions transcendentes de la théorie des nombres, par P. Bachmann, Hadamard et Maillet. Paris, Gauthier-Villars, 1910. 8vo. pp. 193-288. M. 3.00
- ENESTRÖM (G.). Verzeichnis der Schriften Leonhard Euler's. Liefg. 1. Leipzig, Teubner, 1910. 8vo. 2 + 209 pp. M. 10.00
- FERRARI (F.). Die geometrische Lösung der Aufgaben 3ten und 4ten Grades mittels des Lineals und einer festen Kurve 3ter Ordnung mit Rückkehrpunkt oder reellem Doppelpunkte. Münster, 1910. 8vo. 54 pp. M. 2.00
- FLÜKIGER (H.). Die Flächenenteilung des Dreiecks mit Hilfe der Hyperbel. (Diss.) Bern, Wyss, 1910. 8vo. 4 + 51 pp.
- FREUND (E.). Die Brennpunktskreise der Kegelschnittslinien. (Progr.) Pilsen, 1909. 8vo. 10 pp.
- GALLATLY (W.). The modern geometry of the triangle. London, Hodgson, 1910. 8vo. 70 pp. Sewed. 2s. 6d.
- GARRIGOS (V.). Nuove curve (V) e loro proprietà. Roma, Accad. d. Lincei, 1910. 4to. 25 pp.
- GEIGER (K.). Zwei Kurven zweiter Ordnung in zwei-, drei-, und vierpunktiger Berührung. Ergänzungen zum Programm für 1906-07. (Progr.) Landshut, 1910. 8vo. 27 pp.
- GOUBESAT (E.). Cours d'analyse mathématique. Seconde édition, entièrement refondue. Tome premier: Dérivées et différentielles. Intégrales définies. Développement en séries. Applications géométriques. Paris, Gauthier-Villars, 1910. 8vo. 6 + 647 pp. Fr. 20.00
- HAAG. Familles de Lamé, composées de surfaces égales. (Thèse.) Paris, Gauthier-Villars, 1910. 4to. 89 pp.
- HUBER (P.). Die Scyphoide, eine Kurve 4ter Ordnung. (Diss.) Bern, Dürrenmatt-Egger, 1910. 8vo. 4 + 52 pp.
- IST Mathematik Hexerei? Von einem preussischen Schulmeister. Freiburg, Herder, 1910. 8vo. 4 + 68 pp. M. 1.20
- KOPPE. Mathematische Modelle zum Selbstanfertigen. Nordhausen, Koppe, 1910. 9 plates + 11 pp. M. 1.50
- KREIS (H.). Einige Anwendungen der Matricestheorie. Winterthur, Ziegler, 1910. 8vo. 65 pp.
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- MILNE (W. P.). Homogeneous coordinates for use in colleges and schools. London, Arnold, 1910. 8vo. 176 pp. 5s.
- MONTEL (P.). Leçons sur les séries de polynômes à une variable complexe. Paris, Gauthier-Villars, 1910. 8vo. 11 + 130 pp. Fr. 3.50
- MORABITO (G.). Note sur une méthode élémentaire pour établir quelques formules de géométrie analytique plane. Liège, Pholien, 1910. 4to. 15 pp. Fr. 1.50
- PASTOR (J. R.). Correspondencia de figuras elementales. Con aplicación al estudio de las figuras que engendran. Curso 1908-09. Madrid, Hernández, 1910.
- PIESCHEL (F.). Ueber die Krümmung von Flächen in gewöhnlichen Flächenpunkten und die Krümmung der Mittelpunktsflächen $Lw^2 + My^2 + Nz^2 = 1$ insbesondere. (Progr.) Teschen, 1909. 8vo. 32 pp.
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- SCHWENDENWEIN (H.). Ein Satz aus der Zahlentheorie. (Progr.) Klagenfurt, 1910. 8vo. 22 pp.
- TANNERY (J.). Introduction à la théorie des fonctions d'une variable. Tome II: Intégrales définies, développements en séries, langage géométrique, fonctions de variables imaginaires. Avec une note de M. Hadamard. Paris, Hermann, 1910. 8vo. 484 pp. Fr. 15.00

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- BAKER (W. M.) and BOURNE (A. A.). The student's arithmetic. London, Bell, 1910. 8vo. 336 pp. 2s. 6d.
- BARR (T.). Practical mathematics for continuation classes. London, Blackie, 1910. 8vo. 240 pp. 2s.
- BATES (E. L.) and CHARLESWORTH (F.). Practical mathematics and geometry. Specially arranged to meet the revised requirements of the board of education. London, Batsford, 1910. 8vo. 454 pp. 3s.

BAUSCHINGER (J.) und PETERS (J.). Logarithmisch-trigonometrische Tafeln, neu berechnet. Band 1: Tafel der 8-stelligen Logarithmen aller Zahlen von 1-200000. Leipzig, Engelmann, 1910. 8vo. 20 + 368 pp. Cloth. M. 18.50

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BUBALI-FORTI (C.) e RAMORINO (A.). Elementi di algebra, per le scuole medie inferiori. Terza edizione, intieramente rifatta. Torino, Gallizio, 1910. 8vo. 7 + 142 pp. L. 1.75

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DAVISSON (S. C.). College algebra. New York, Macmillan, 1910. 12mo. 243 pp. Half leather. \$1.40

DONADT (A.). See AUTENHEIMER (F.).

DUPUIS (J.). Tables de logarithmes à cinq décimales. 30e édition. Paris, Hachette, 1910. 16mo. 4 + 230 pp. Fr. 2.00

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ENRIQUES (F.) e AMALDI (U.). Nozioni di geometria, ad uso delle scuole complementari. Bologna, Zanichelli, 1910. 16mo. 7 + 195 pp. L. 1.80

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MCGIFFERT (J.). *Mathematical short cuts.* Troy, N. Y., Allen, 1910. 16mo. 31 pp. Paper. \$0.25

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- PICKEN (D. K.). The theory of elementary trigonometry. London, Whitcombe, 1910. 8vo. 2s. 6d.
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- GOUARD (E.) et HIERNAUX (G.). Cours élémentaire de mécanique industrielle. Compléments de cinématique. Résistance des matériaux. Moteurs hydrauliques et machines à vapeur. Tome 2. Paris, Dunod, 1910. 8vo. 363 pp.
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- SENNETT (R.) and ORAM (H. J.). The marine steam engine. 10th edition. London, Longmans, 1910. 8vo. 510 pp. 21s.
- SOUTHALL (J. P. C.). The principles and methods of geometrical optics, especially as applied to the theory of optical instruments. New York, Macmillan, 1910. 8vo. 23 + 628 pp. Cloth. 85.50
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THE OCTOBER MEETING OF THE AMERICAN
MATHEMATICAL SOCIETY.

THE one hundred and fiftieth regular meeting of the Society was held in New York City on Saturday, October 29, 1910. The attendance at the morning and afternoon sessions included the following forty-three members:

Mr. F. W. Beal, Professor G. D. Birkhoff, Professor E. W. Brown, Mr. R. D. Carmichael, Dr. Emily M. Coddington, Professor F. N. Cole, Dr. G. M. Conwell, Dr. Elizabeth B. Cowley, Professor E. S. Crawley, Professor L. P. Eisenhart, Professor H. B. Evans, Professor H. B. Fine, Professor T. S. Fiske, Professor W. B. Fite, Professor O. E. Glenn, Professor C. C. Grove, Professor C. O. Gunther, Professor H. E. Hawkes, Dr. Frank Irwin, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Mr. W. C. Krathwohl, Dr. N. J. Lennes, Professor J. H. MacLagan-Wedderburn, Dr. H. F. MacNeish, Dr. H. H. Mitchell, Professor Richard Morris, Professor G. D. Olds, Professor W. F. Osgood, Dr. H. W. Reddick, Mr. L. P. Siceloff, Dr. L. L. Silverman, Mr. C. G. Simpson, Professor D. E. Smith, Professor P. F. Smith, Mr. W. M. Smith, Professor Virgil Snyder, Professor H. D. Thompson, Dr. M. O. Tripp, Professor Oswald Veblen, Mr. H. E. Webb, Professor H. S. White.

Ex-President W. F. Osgood occupied the chair at the morning session, Ex-President H. S. White and Professor Kasner at the afternoon session. The Council announced the election of the following persons to membership in the Society: Dr. G. A. Campbell, American Telephone and Telegraph Company; Mrs. E. B. Davis, Nautical Almanac Office; Professor C. W. Emons, Simpson College; Professor H. C. Feemster, York College; Mr. R. R. Hitchcock, University of North Dakota; Mr. W. J. Montgomery, University of Michigan; Professor C. C. Morris, Ohio State University; Mr. H. S. Newcomer, University of Wisconsin; Professor A. D. Pitcher, University of Kansas; Professor George Rutledge, Georgia School of Technology. Four applications for membership in the Society were received.

The list of nominations for officers and other members of the

Council to be placed on the official ballot for the annual meeting was adopted. Provision was made for the appointment of a committee to audit the Treasurer's accounts. The Council decided to undertake the republication of the Evanston Colloquium Lectures and to place them on sale at a nominal price.

The following papers were read at the October meeting :

(1) Professor G. A. MILLER: "The group generated by two conjoints."

(2) Professor O. E. GLENN: "The conditions that the p -ary form of order m be a perfect m th power."

(3) Professor EDWARD KASNER: "A second converse of the theorem of Thomson and Tait."

(4) Dr. L. L. SILVERMAN: "Generalized definitions of the sum of divergent series."

(5) Dr. H. H. MITCHELL: "Note concerning a collineation group in n variables."

(6) Mr. R. D. CARMICHAEL: "Mixed equations and their analytic solutions" (preliminary communication).

(7) Professor G. A. MILLER: "Groups generated by two operators satisfying two conditions."

In Professor Miller's absence his papers were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. One of the most useful theorems proved by Jordan in his first published paper is that with every regular group H of degree n there is associated another regular group H' on the same letters such that every substitution of each of these two groups is commutative with every substitution of the other. The groups H and H' are known as conjoints, and the group G generated by H and H' has received considerable attention. In particular, Maillet observed that the necessary and sufficient condition that G be primitive is that H is a simple group. If H is abelian, H and H' are identical. This trivial case is excluded in what follows. Maillet considered also in his grand prix memoir of 1893 the class of G when H is a simple group of composite order and observed that an inferior limit is $\frac{4}{3}n$. This limit is also given in the *Encyclopédie des Sciences mathématiques*, tome 1, volume 1, page 593. Professor Miller proves that the inferior limit of the class of G when H is simple is really $\frac{11}{3}n$, and that the simple group of order 60 is the only one which gives such a low limit. When H is com-

posite, the inferior limit is $\frac{1}{2}n$. Moreover, G is always simply transitive and each of its substitutions which omits at least one letter must omit a number of letters which is divisible by its order. When G is primitive we thus have an infinite category of simply transitive primitive groups in which every substitution which omits one letter must omit at least two letters. Professor Cole called attention to a simply transitive primitive group of degree 9 which involves some substitutions with this property and observed that no such group exists whose degree is less than 9.* Professor Miller's paper has been offered for publication in *Prace Matematyczno-Fizyczne*.

2. By making use of certain results on factorable quantics which he has published previously, Professor Glenn proves that the necessary and sufficient conditions that a p -ary m -ic be a perfect m th power consist (1) in the

$$\binom{m+p-1}{p-1} - m(p-1) - 1$$

conditional relations which exist among the coefficients of the form in order that it be factorable into linear factors, and (2) the identical vanishing of $p-1$ binary Hessian determinants. The explicit forms of these Hessians, for p and m both general, are developed.

3. The purely geometric part of the theorem of Thomson and Tait suggested the converse question discussed by Professor Kasner in a former paper (*Transactions*, April, 1910). The present converse relates to the distribution of velocities along an initial base surface. In general the only possible distributions leading to normal congruences of trajectories are those for which the sum of the kinetic and potential energies is constant. The exceptional cases are limited and are discussed completely for central forces.

4. All of the definitions hitherto proposed for the sum of a divergent series have aimed to cover an increasingly wide range of series that will have a meaning. It then becomes desirable to extend theorems involving series to the case where the series diverge. In such cases, however, the extension of the theorem

* BULLETIN, vol. 2 (1893), p. 258.

in question to any new definition would require either a different proof, or at least a restatement of the proof given for previous definitions. To avoid the necessity for different proofs or for restatements of the same proof, Dr. Silverman states a definition for the sum of a series which not only contains all of the well-known definitions of summability, but includes all definitions of a certain type which are subject to the following requirements:

(i) The generalized sum must exist and be equal to the ordinary sum whenever the series converges.

(ii) If a term u_0 is dropped from a series with generalized sum s , the resulting series has a generalized sum, which is $s - u_0$.

(iii) If two series with s and t for generalized sums are added term by term, the resulting series has a generalized sum which is $s + t$.

A series $u_1 + u_2 + \dots + u_n + \dots$ is defined to be *evaluable** if there exists a set of functions $a_i(\alpha)$, where (α) is any assemblage with a cluster point at $+\infty$, such that the following conditions are fulfilled when $s_n = u_1 + u_2 + \dots + u_n$:

$$1) \quad a_i(\alpha) \geq 0, \quad 2) \quad \lim_{\alpha \rightarrow \infty} a_i(\alpha) = 0 \text{ for every } i,$$

$$3) \quad \lim_{n \rightarrow \infty} \sum_{i=1}^n a_i(\alpha) = 1, \quad 4) \quad \lim_{\alpha \rightarrow \infty} \lim_{n \rightarrow \infty} \sum_{i=1}^n a_i(\alpha) s_i = s,$$

$$5) \quad \lim_{\alpha \rightarrow \infty} \lim_{n \rightarrow \infty} \sum_{i=1}^n a_i(\alpha) u_i = 0.$$

This definition satisfies all of the requirements above. It includes, too, all of the definitions of summability which also satisfy these requirements. For $a_i(\alpha) = e^{-\alpha} \cdot \alpha^i / i!$, we obtain Borel's mean summability.† By putting

$$a_i(\alpha) \equiv a_i(n) \equiv \frac{r(r-1) \dots (r+n-i)}{r-i+1} \bigg/ \frac{(r+1) \dots (r+n)}{n!},$$

we have Cesàro's summability of order r . Other definitions

* It is convenient to have a new name for series which have a value according to this general definition; a series will then be called *summable*, if it has a value for a particular set of $a_i(\alpha)$.

† Note: Borel's integral definition cannot be included since it fails to satisfy the second requirement. To meet this difficulty Borel introduces absolute summability; but this definition fails to satisfy the first requirement. See Bromwich, *An Introduction to the Theory of Infinite Series*, pp. 273 and 279.

may of course be given by specializing the $a_i(\alpha)$. Thus, e. g.,

$$a_i(\alpha) \equiv a_i(n) \equiv \frac{1}{i \log n}$$

gives

$$s = \lim_{n \rightarrow \infty} \left(s_1 + \frac{s_2}{2} + \dots + \frac{s_n}{n} \right) / \log n,$$

which has proved useful in connection with Dirichlet series.*

The paper will be published in the *Studies of the University of Missouri*.

5. In a paper concerning commutative collineations, in a recent number of the *Annals of Mathematics*, Professor W. B. Fite has called attention to an abelian group of order n^2 in n variables. This group may be generated by the two transformations

$$S: x'_i = \omega^i x_i, \quad T: x'_i = x_{i+1},$$

where i is to be taken modulo n and ω is an n th root of unity.

This group of order n^2 is invariant under a group of order

$$n^s \left(1 - \frac{1}{p^s} \right) \left(1 - \frac{1}{q^s} \right) \dots,$$

(where $n = p^s q^s \dots$) which was discovered by Klein. If n is prime, this group is of order $(p+1)p^s(p-1)$, and it is proved by Dr. Mitchell to be simply isomorphic with the group of all transformations of the form

$$y' = a_1 y + a_2 z + a_3, \quad z' = b_1 y + b_2 z + b_3,$$

where the coefficients are marks of the $GF(p)$ and

$$a_1 b_2 - a_2 b_1 = 1.$$

6. In this preliminary communication Mr. Carmichael points out the existence in general of an infinite number of solutions of the mixed system of equations

$$D_i f_i(x) = \sum_{j=1}^n \phi_{ij}(x) f_j(x) \quad (i = 1, \dots, n),$$

where

$$\phi_{ij}(x) = a_{ij} x^{-s} + a'_{ij} x^{-s} + \dots \quad (|x| \geq R),$$

* Ruiz: *Comptes Rendus*, July 5, 1909.

D_i ($i = 1, \dots, k$) denoting D the symbol of differentiation, and D_i ($i = k + 1, \dots, n$) denoting Δ the symbol of differences.

7. The first part of Professor Miller's second paper is devoted to a consideration of all the possible groups which can be generated by two operators s_1, s_2 when these operators satisfy two conditions of the form

$$s_1^{\alpha_1} s_2^{\alpha_2} s_1^{\alpha_3} \dots = 1, \quad s_1^{\beta_1} s_2^{\beta_2} s_1^{\beta_3} \dots = 1,$$

where each one of a set of consecutive exponents is unity while each of the other exponents is zero. He proved the following theorem: If identity can be obtained in two ways by forming the continued product with s_1, s_2 taken alternately as factors, then the largest group G generated by s_1, s_2 is of finite order except when the number of factors in both products is even, or when the number of these factors is the same odd number in both products and the same operator occurs an odd number of times in each. When the total number of times that each factor appears in the two products is the same and the number of factors in each product is odd then G is a cyclic group whose order is this total number. Among other theorems proved in this paper is the following: If two operators satisfy the condition $(s_1 s_2)^\alpha = (s_2 s_1)^\beta$, α and β being relatively prime, the order of $s_1 s_2$ may have an arbitrary value n , which is prime to α and β , and hence $s_1 s_2 = (s_2 s_1)^\gamma$. The order of s_2 is an arbitrary multiple of the exponent e to which γ belongs modulo n and hence the order of G is a multiple of en . All such groups are solvable.

F. N. COLE,
Secretary.

THE KÖNIGSBERG MEETING OF THE DEUTSCHE MATHEMATIKER-VEREINIGUNG.

THE twenty-first annual meeting of the Deutsche Mathematiker-Vereinigung was held at Königsberg, Prussia, September 18–24, 1910, in affiliation with the eighty-second convention of the Society of German naturalists and physicians.

A general reception for all the sections was held at the Festhalle of the Tiergarten on the evening of September 18; the division into sections and the organization of each took place the following morning. Six sessions of the Vereinigung

were held, under the chairmanship of Professors Hilbert, Meyer, v. Mangoldt, and Schoenflies.

The following papers were read:

(1) R. FUETER, Basel, "Fields of classes of complex multiplication, and its influence on the development of the theory of numbers."

(2) D. HILBERT, Göttingen, "On diophantine differential equations."

(3) A. HAAR, Göttingen, "Illustrations of and additions to the principles explained by Professor Hilbert."

(4) P. KOEBE, Göttingen, "Concerning conformal representation of multiply connected regions."

(5) E. MÜLLER, Vienna, "Some groups of theorems regarding oriented circles in the plane."

(6) F. MEYER, Königsberg, "A generalization of the concept of curvature."

(7) H. LIEBMANN, Leipzig, "On elementary constructions in non-euclidean space."

(8) L. BIEBERBACH, Göttingen, "On the group of motions in euclidean geometry."

(9) F. ENGEL, Greifswald, "A generalization of the infinitesimal parallel transformation."

(10) E. PAPPERITZ, Freiberg, "On drawing in space" (with the exhibition of apparatus for kinodiaphragmatic projection).

(11) A. SCHOENFLIES, Königsberg, "On the concept of definitions."

(12) A. WITTING, Dresden, "Communications concerning some manuscripts of Newton."

(13) W. v. IGNATOWSKI, Berlin, "The principle of relativity."

(14) F. MEYER, Königsberg, "On the theory of rotations."

(15) O. TOEPLITZ, Göttingen, "Some applications of the theory of an infinite number of variables."

(16) H. WEYL, Göttingen, "Developments in series, and the integral representation of arbitrary functions."

(17) A. SOMMERFELD, Munich, "Development of Green's function by means of characteristic functions."

(18) W. KREBS, Grossflotbeck, "New developments in the spectral photography of the sun."

(19) W. KREBS, Grossflotbeck, "Solar activity as volcanic action. Foundations of the theory."

In addition to the preceding papers, five others were read

before the joint meeting of the sections of mathematics, physics, and geodesy. They were devoted to an estimate of the influence of Bessel and of F. Neumann. The business meeting of the society occupied the entire morning session of September 21. Professor Schur, of Strassburg, was elected president for the coming year. Professors Dyck, of Munich, and Lilienthal, of Münster, were elected to the Council. The next meeting of the Vereinigung will be held at Carlsruhe in September, 1911.

Abstracts of some of the papers are given below, the numbers corresponding to those in the list above.

1. Kummer recognized that in the theory of algebraic numbers the field of the investigation must be extended until all the properties can be recognized. Apart from the concept of ideals, the Kronecker theory of complex multiplication forms the most important recent extension, since it brings in the aid of another hyperfield in order to develop the theory of numbers of an imaginary quadratic field. Besides the known concepts of field, rings, orders, moduli, it is also necessary to consider regions in which numbers of the set remain such by multiplication and division. Professor Fueter has studied such sets, which he here calls rays. Ideals are divided into classes of rays. If any equation in a field K is given, a definite ray belongs to it, connecting it with the known theory. Equations are made of exponential functions $e^{2\pi iz}$, z being rational. If we consider the law of factoring of the prime ideals of K in the hyperfield, we obtain the law of reciprocity. By this means we may calculate the number of classes in the field. For the case in which the ray is the field itself, Hilbert and Furtwängler have constructed the associated equations by arithmetic means.

5. Professor Müller emphasized the importance of cyclographical representation for geometric investigations and gave some illustrations. He obtained, for example, from the theorem: if of the four pairs of tangent planes which can be passed through the sides of a simple quadrilateral to a curve of class 2, four planes pass through a point, then the remaining ones also pass through a point, by means of cyclographic representation, a theorem regarding directed circles and straight lines, of which a number of theorems of Steiner (*Crelle*, volume 11 (1831), pages 117–129) are special cases. He showed further that the Feuerbach theorem, dualized in space, becomes:

if $t_1, t'_1; t_2, t'_2; t_3, t'_3$ are pairs of points of a quadratic cone, the lines joining the points of a pair passing through a fixed point, then the planes $(t_1 t'_1), (t_2 t'_2), (t_3 t'_3)$ cut the cone in four curves, touched by a plane section of the surfaces. In this way only can the theorems in the plane express their whole meaning.

9. If the expression $ds = \omega(x, y, y')dx$ is taken for element of arc, the equation $ds = \delta t$ defines an infinitesimal circle of radius δt and center x, y . If these circles are constructed for all points of a curve $y = f(x)$, there results, as envelope, an infinitesimally different curve, obtained from the formula by an infinitesimal contact transformation. A particular case is the infinitesimal parallel transformation of the euclidean plane. If now an arbitrary contact transformation is performed upon the element of arc, a new form is obtained,

$$ds = [\alpha(x, y, y') + \beta(x, y, y')y'']dx,$$

and the contact transformation previously defined is invariant with regard to contact transformations. A direct definition of this transformation is possible, based upon the new form of the element of arc. The purpose of Professor Engel's paper was to supply such a definition.

10. Professor Papperitz discussed the operations of drawing in space, presenting his apparatus for kinodiaphragmatic projection. The apparatus serves the purpose of representing geometric figures by means of projection of movable slits and in fact makes it possible to represent sharply and exactly the most diverse plane and space curves, and to illustrate changes of form produced by changing constants. The images in space are visible from every side. The process is patented; it should materially aid in the instruction in geometry. A prospectus will soon be published by Teubner. The fundamental idea is the following: The three necessary provisions for the representation are 1) adjustable source of light, 2) the original curve, 3) screen; the first is retained, but the others may be replaced by new apparatus, which will actually generate the image. These generators are constructed by mathematical rules, and select rays of light, part being reflected, part absorbed. Illustrations of systems of conics and of curves or quadrics were exhibited.

12. Dr. Witting reported on the preservation and the condition of the manuscripts of Newton in the library of Cambridge University, and gave some details of unpublished papers on mathematics, relating particularly to early investigations on the calculus.

At the joint meeting Professor Wangerin considered Franz Neumann (1798–1895) first in connection with mathematical physics, in which he was the first and for a long time the ablest German representative; he then discussed his achievements as an academic teacher, and finally as a mathematician.

A publication of 1826 on the Apollonius tactic problem is not well known; while the method is similar to that of French writers of the time, the treatment is more general in that contact is replaced by intersection at given angles, and both cones and spheres are discussed.

There are three purely arithmetic papers on the theory of harmonic functions. In the first (1828), the problem is solved of expanding an unknown function of the latitude and longitude based upon observations into a finite series of Laplace functions. The second (1848) discusses the application of harmonic functions to problems in the theory of the potential on surfaces of revolution, in particular to determine the distribution of terrestrial magnetism. In this paper the Neumann integral first appears. The third is an extensive essay (1878) concerning adjoint spherical harmonics in which the secondary index is larger than the principal index. A method is perfected by which the product of two spherical harmonics can be expanded in a series of harmonic functions. In many other investigations analytical skill of high order is noticeable, particularly in crystallography. Least squares were first applied to harmonize crystallographic measurements. Experimental researches in specific heat were based upon analytic developments (1830–1834). In optics new roads were broken, particularly in double refraction (1832), reflection of metals (1837), color of biaxial crystals in polarized light (1834), and many others, all based on mathematical foundations. Since 1840 most of Neumann's investigations were in connection with electricity.*

* See Wangerin: "Franz Neumann, sein Wirken als Forscher und Lehrer," Braunschweig, Vieweg, 1907.

ON THE SADDLEPOINT IN THE THEORY OF MAXIMA AND MINIMA AND IN THE CALCULUS OF VARIATIONS.

BY PROFESSOR R. G. D. RICHARDSON.

(Read before the American Mathematical Society, April 30, 1910.)

Introduction.

LAGRANGE has shown that the problem of determining a function $y(x)$ which satisfies the boundary conditions

$$(1) \quad y(0) = y(1) = 0$$

and the integral condition

$$(2) \quad \int_0^1 g(x, y, y') dx = 0$$

and which minimizes the integral

$$(3) \quad \int_0^1 f(x, y, y') dx$$

is equivalent, as far as the first variation is concerned, to the problem of minimizing the integral

$$(4) \quad \int_0^1 (f + \lambda g) dx,$$

the function being subject to no isoperimetric condition. The two constants of integration and the isoperimetric constant λ of the Lagrange differential equation

$$\frac{d}{dx} (f_y + \lambda g_y) + f_x + \lambda g_x = 0$$

which furnishes the solution are determined from the conditions (1) and (2). On the other hand it is possible to consider the problem of minimizing the integral (4) subject only to the boundary condition (1), in which case the minimum is obviously a function of the parameter λ . The determination of that value of λ which maximizes this minimum is a saddlepoint problem. The methods of this paper suffice to show that the first necessary condition for a solution $y(x)$ of this problem is identical with that for a solution of the foregoing isoperimetric problem.

Of a similar nature is the problem of minimizing the integral (3) for those values of $y(x)$ which satisfy the integral condition

$$\int_0^1 \{g(x, y, y') + \kappa h(x, y, y')\} dx = 0.$$

The minimum is a function $m(\kappa)$ of the parameter κ . The maximizing of $m(\kappa)$, that is, the determination of $y(x, \kappa)$ in such a way that it furnishes a maximum of a minimum is again a saddlepoint problem. It will be shown (§ 2) that the first necessary condition for a solution is identical with that for a solution of the problem of finding a minimum of (3) for those functions which satisfy the conditions

$$\int_0^1 g dx = 0, \quad \int_0^1 h dx = 0.$$

In the theory of maxima and minima there are also saddlepoint problems and related isoperimetric problems quite analogous to the foregoing and their treatment being naturally simpler is first considered in the discussion. In § 3 is given an application to an example which arises in the study of n self-adjoint linear differential equations of the second order containing n parameters.

The corresponding theory for the case of more general isoperimetric conditions is reserved for a later discussion.

§ 1. *An Extremum of an Extremum in the Theory of Maxima and Minima.*

Any solution of the problem of minimizing a function $f(x_1, \dots, x_n)$ for those values of the variables which are subject to the relation $g(x_1, \dots, x_n) = 0$ must satisfy the equations

$$(5) \quad \frac{\partial f}{\partial x_i} + \lambda \frac{\partial g}{\partial x_i} = 0 \quad (i = 1, \dots, n),$$

$$(6) \quad g(x_1, \dots, x_n) = 0.$$

In case a solution exists, the variables $\lambda, x_1, \dots, x_n$ are determined from the $n + 1$ equations (5) and (6).

Related to the preceding problem is the following: The minimum of the function $f(x_1, \dots, x_n) + \lambda g(x_1, \dots, x_n)$ is a function $m(\lambda)$ of the parameter λ ; it is required to maximize this minimum considered as a function of λ . In case a minimum

exists, the variables $x_1, \dots, x_n$ are determined from the equations

$$(7) \quad \frac{\partial f}{\partial x_i} + \lambda \frac{\partial g}{\partial x_i} = 0 \quad (i = 1, \dots, n).$$

To determine the maximum $m(\lambda)$ it is necessary to add to (7) the condition

$$(8) \quad \frac{\partial}{\partial \lambda} (f + \lambda g) = \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial \lambda} + \lambda \frac{\partial g}{\partial x_i} \frac{\partial x_i}{\partial \lambda} \right) + g = 0.$$

That the equations (7) and (8) are equivalent to (5) and (6) may be readily seen by multiplying the equations (7) by $\partial x_1 / \partial \lambda, \dots, \partial x_n / \partial \lambda$ respectively, and subtracting their sum from (8).

This result may be stated as follows: *The minimum of $f(x_1, \dots, x_n) + \lambda g(x_1, \dots, x_n)$, ($n > 1$), for all values of $x_1, \dots, x_n$ is a function of the parameter λ which when maximized gives the same constant as the minimum of f for those values of the variables which satisfy the relation $g = 0$.*

The minimum of the function $f(x_1, \dots, x_n)$ for those values of the variables which satisfy the relations $g(x_1, \dots, x_n) = 0$, $h(x_1, \dots, x_n) = 0$ is found by means of the equations

$$(9) \quad \frac{\partial f}{\partial x_i} + \lambda \frac{\partial g}{\partial x_i} + \mu \frac{\partial h}{\partial x_i} = 0 \quad (i = 1, \dots, n),$$

$$(10) \quad g(x_1, \dots, x_n) = 0, \quad h(x_1, \dots, x_n) = 0.$$

The minimum of the function $f(x_1, \dots, x_n)$ for those values of the variables which satisfy the relation

$$g(x_1, \dots, x_n) + \kappa h(x_1, \dots, x_n) = 0$$

is a function of the parameter κ which when maximized must satisfy the equations

$$(11) \quad \frac{\partial f}{\partial x_i} + \lambda \left(\frac{\partial g}{\partial x_i} + \kappa \frac{\partial h}{\partial x_i} \right) = 0,$$

$$(12) \quad g(x_1, \dots, x_n) + \kappa h(x_1, \dots, x_n) = 0,$$

$$(13) \quad \frac{\partial f}{\partial \kappa} = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial \kappa} = 0.$$

Proceeding now to show that the sets of equations (9), (10) and (11), (12), (13) are equivalent, we multiply the equations (11)

by $\partial x_1/\partial \kappa$, $\dots$, $\partial x_n/\partial \kappa$ respectively, from the sum subtract (13), and obtain

$$\lambda \sum_{i=1}^n \left(\frac{\partial g}{\partial x_i} \frac{\partial x_i}{\partial \kappa} + \kappa \frac{\partial h}{\partial x_i} \frac{\partial x_i}{\partial \kappa} \right) \equiv \lambda \left(\frac{dg}{d\kappa} + \kappa \frac{dh}{d\kappa} \right) = 0.$$

Disregarding the case where $\lambda = 0$,* this becomes

$$(14) \quad \frac{dg}{d\kappa} + \kappa \frac{dh}{d\kappa} = 0.$$

On the other hand by differentiating (12) with regard to κ , it follows that

$$(15) \quad \frac{dg}{d\kappa} + \kappa \frac{dh}{d\kappa} + h = 0.$$

and from (14), (15) and (12) that

$$h(x_1, \dots, x_n) = 0, \quad g(x_1, \dots, x_n) = 0.$$

On setting $\lambda\kappa = \mu$ the complete equivalence of the two sets of equations is established and the first necessary conditions† for the two problems are seen to be identical.

Since these methods of proof are perfectly general, the result may be enunciated as follows: *The minimum of the function $f(x_1, \dots, x_n)$ for those values of the variables which satisfy the relation*

$$g(x_1, \dots, x_n) + \sum_{i=1}^m \kappa_i h_i(x_1, \dots, x_n) = 0 \quad (m+1 < n)$$

is a function of $\kappa_1, \dots, \kappa_m$. The first necessary condition for a maximum of this function is identical with that for a minimum of $f(x_1, \dots, x_n)$ for those values of $x_1, \dots, x_n$ which satisfy the relations $g = 0, h_1 = 0, \dots, h_m = 0$.

§ 2. An Extremum of an Extremum in the Calculus of Variations.

The problem of determining a function $y(x)$ which satisfies the boundary conditions

$$(16) \quad y(0) = y(1) = 0$$

and the integral conditions‡

* In case $\lambda = 0$, the solution is determined from the equations $\partial f/\partial x_i = 0$ ($i=1, \dots, n$). The minimum is then independent of κ and the result we are seeking to establish is self-evident.

† On setting $g_1 = g - c$, the case of the more general integral condition $\int_0^1 g dx = c$ is reduced to $\int_0^1 g_1 dx = 0$.

$$(17) \quad \int_0^1 g(x, y, y') dx = 0, \quad \int_0^1 h(x, y, y') dx = 0$$

and which minimizes the integral

$$(18) \quad \int_0^1 f(x, y, y') dx$$

leads to the Lagrange equation

$$(19) \quad \frac{d}{dx} (f_y + \lambda g_y + \mu h_y) - (f_y + \lambda g_y + \mu h_y) = 0.$$

On the other hand the minimum of the integral (18) for those functions $y(x)$ which satisfy the boundary conditions (16) and the integral condition

$$(20) \quad \int_0^1 \{g(x, y, y') + \kappa h(x, y, y')\} dx = 0$$

is furnished by the equation

$$\frac{d}{dx} \{f_y + \lambda(g_y + \kappa h_y)\} - \{f_y + \lambda(g_y + \kappa h_y)\} = 0.$$

The condition that this minimum be maximized as a function of the parameter κ is identical with that of finding a maximum of the function

$$\phi(\epsilon_1) = \int_0^1 f(x, \bar{y} + \epsilon_1 \eta, \bar{y}' + \epsilon_1 \eta') dx$$

subject to the isoperimetric condition

$$\psi(\epsilon_1, \epsilon_2) = \int_0^1 \{g(x, \bar{y} + \epsilon_1 \eta, \bar{y}' + \epsilon_1 \eta') + (\bar{\kappa} + \epsilon_2) h(x, \bar{y} + \epsilon_1 \eta, \bar{y}' + \epsilon_1 \eta')\} dx = 0,$$

where $\bar{y}(x)$ and $\bar{\kappa}$ are the values of the function and parameter which furnish the extremum of the extremum, η an admissible variation arbitrarily chosen, and ϵ_1, ϵ_2 infinitesimals. On differentiating $\phi + \lambda\psi$ with regard to ϵ_1, ϵ_2 respectively and equating the results to zero for $\epsilon_1 = \epsilon_2 = 0$, one obtains the necessary conditions

$$(21) \quad \int_0^1 [\{f_y + \lambda(g_y + \bar{\kappa} h_y)\} \eta' + \{f_y + \lambda(g_y + \bar{\kappa} h_y)\} \eta]_{y=\bar{y}} dx = 0,$$

$$(22) \quad \int_0^1 h(x, \bar{y}, \bar{y}') dx = 0.$$

Since η is an arbitrary function vanishing at the points $x = 0$, $x = 1$, by applying the product integration usual in such cases, it is readily shown that on setting $\mu = \kappa\lambda$, equation (21) is equivalent to (19). Since equations (20) and (22) are equivalent to (17), it follows that the first necessary condition which a solution of this saddlepoint problem must satisfy is identical with that in the ordinary isoperimetric problem with two integral conditions.

The method here employed may be applied to the other problem proposed in the introduction; it admits also of immediate generalization to functions of several variables subject to integral conditions involving several parameters.

§ 3. Application to an Example.

As an application of the theory of the preceding section let us consider the problem* of finding functions $u_1(x)$, $u_2(x)$ which satisfy the boundary conditions

$$(23) \quad u_1(0) = u_1(1) = 0, \quad u_2(0) = u_2(1) = 0$$

and the integral condition

$$(24) \quad \int_0^1 [l_1(x)u_1^2(x) + l_2(x)u_2^2(x) - 1 + \kappa \{r_1(x)u_1^2(x) + r_2(x)u_2^2(x)\}] dx = 0,$$

and which minimize the integral

$$D(u_1, u_2) = \int_0^1 \{p_1(x)u_1'^2(x) + p_2(x)u_2'^2(x) - q_1(x)u_1^2(x) - q_2(x)u_2^2(x)\} dx,$$

where $p_i > 0$, $q_i \leq 0$, l_i , r_i , ($i = 1, 2$), are analytic functions of x in the interval $(0, 1)$. In order that a minimum† exist for

* Other aspects of this problem are considered by the author in a paper to be published in the *Mathematische Annalen*.

† That a minimum exists for all values of κ and is equal to the smallest (say λ_1) of the positive characteristic numbers (Eigenwerte) $\lambda_1, \lambda_2, \dots$; $\lambda'_1, \lambda'_2, \dots$ of the Lagrange equations

$$(A) \quad (p_i u_i')' + q_i u_i + \lambda (l_i + \kappa r_i) u_i = 0 \quad (i = 1, 2)$$

may be proved as follows: Consider the identities

$$p_i u_i'^2 - q_i u_i^2 - \lambda (l_i + \kappa r_i) u_i^2 = \left(\frac{p_i y_i' u_i^2}{y_i} \right)' + p_i y_i^2 \left(\frac{u_i}{y_i} \right)^2 - \frac{u_i^2}{y_i} [(p_i y_i')' + q_i y_i + \lambda (l_i + \kappa r_i) y_i] \quad (i = 1, 2),$$

various values of the parameter κ it is necessary to exclude the case that both of the functions l_1, l_2 are positive throughout the whole interval and also the case that both the functions r_1, r_2 are positive or both negative throughout the interval. (For convenience of notation in the following discussion it is assumed that r_1 takes both signs in the interval.)

If then it is possible to show that the minimum $m(\kappa)$ of $D(u_1, u_2)$ considered as a function of κ possesses a maximum, it follows from the preceding theory that the functions $u_1(x), u_2(x)$ which furnish this maximum of a minimum, furnish also a minimum for the integral $D(u_1, u_2)$ subject to the boundary conditions (23) and the integral conditions

$$\int_0^1 \{l_1(x)u_1^2 + l_2(x)u_2^2\} dx = 1, \quad \int_0^1 \{r_1(x)u_1^2 + r_2(x)u_2^2\} dx = 0.$$

In order to show that this maximum exists, we note that $D(u_1, u_2)$ can be zero only if $u_1 \equiv 0, u_2 \equiv 0$. Since these values will not satisfy (24), a constant $c > 0$ may be chosen such that for values of k within a restricted interval $m(k) > c$. Since m is a continuous analytic function of κ it follows that a maximum exists if it can be shown that $m(\infty) = 0, m(-\infty) = 0$. With this in view let us consider the function pair $u_1(x) \equiv u_1(x), u_2(x) \equiv 0$. The condition (24) may then be written

where $u_1(x), u_2(x)$ are any two functions which vanish at the end points $x=0, x=1$. If $\lambda_0 < \lambda_1$, we must show that $D(u_1, u_2) > \lambda_0$ for all functions $u_1(x), u_2(x)$ which satisfy (24). The functions $y_1(x), y_2(x)$ are chosen to be solutions of the differential equations

$$(p_i y_i')' + q_i y_i + \lambda_0(l_i + \kappa r_i)y_i = 0 \quad (i=1, 2)$$

which vanish neither within the interval nor at the end points. (This is possible since λ_0 is smaller than either of the characteristic numbers λ_1, λ_1' corresponding to those solutions of the differential equations (A) which vanish at the end points. See proof by author, *Mathematische Annalen*, vol. 68, No. 2, 1910.) On setting $\lambda = \lambda_0$ in the identities, integrating and adding, we have

$$\int_0^1 \Sigma(p_i u_i'^2 - q_i u_i^2) dx - \lambda_0 \int_0^1 \Sigma(l_i + \kappa r_i) u_i^2 dx = \int_0^1 \Sigma p_i y_i^2 \left(\frac{u_i}{y_i}\right)'^2 dx.$$

Since $p_i(x)$ and $p_2(x)$ are positive, this may be written $D(u_1, u_2) - \lambda_0 > 0$. That $D(u_1, u_2)$ takes the value λ_1 may be seen by setting $\lambda = \lambda_1$ in the first identity and integrating. For $u_2(x) \equiv 0$ this gives $D(u_1, u_2) - \lambda_1 = 0$ (since y_1 can be chosen equal to u_1). It may be noted that in case the characteristic numbers λ_1 and λ_1' are not equal, in order to furnish a maximum, one of the functions $u_1(x), u_2(x)$ must be identically zero. In case however that $\lambda_1 = \lambda_1'$ the maximum can be furnished by the function pairs (1) $u_1(x) \neq 0, u_2(x) = 0$, (2) $u_1(x) \equiv 0, u_2(x) \neq 0$, or (3) $u_1(x) \neq 0, u_2(x) \neq 0$.

$$(25) \quad \int_0^1 \left\{ \frac{l_1(x)}{\kappa} + r_1(x) \right\} u_1^2(x) dx = \frac{1}{\kappa}.$$

By taking κ large enough it is possible to choose $u_1(x)$ (in an infinite number of ways) such that equation (25) is satisfied and $|u_1(x)| < \epsilon$, $|u_1'(x)| < \epsilon$, where ϵ is an arbitrarily small positive number. For this value of $u_1(x)$ (and $u_2(x) \equiv 0$), $D(u_1, u_2) < C\epsilon^2$, where C is a constant such that $C > p_1(x) - q_1(x)$. The value of κ is then so chosen that $m(\kappa)$ is less than $C\epsilon^2$. In a similar manner it may be shown that $m(-\infty) = 0$.

BROWN UNIVERSITY, PROVIDENCE,
September, 1910.

NOTE ON IDENTITIES CONNECTING CERTAIN INTEGRALS.

BY DR. LOUIS INGOLD.

(Read before the American Mathematical Society, September 7, 1910.)

BECAUSE of the general nature of the symbols and symbolic products used in the symbolic invariant theory, it is possible to apply the formulas of this theory in various special fields. In the present note the theory is employed to obtain relations connecting integrals of functions constructed out of a linearly independent set.

1. We shall be interested in functions of n parameters $u_1, u_2, \dots, u_n$ and functions of one or more real variables $x, x_1, x_2, \dots, y, y_1, y_2, \dots$, etc., restricted to a definite interval, say, $0 \leq x \leq 1$. In order to distinguish readily between those quantities which are constant or functions of the parameters $u_1, \dots, u_n$ only, and those which are functions of one or more of the variables $x, x_1, x_2, \dots$, etc., the latter will always be denoted by black faced letters; the others by letters in ordinary type; thus a denotes a function of a variable x defined for values of x on the interval $0 \dots 1$, while α denotes a function of the u_i only or a constant. The partial derivatives $\partial a / \partial u_i, \partial \alpha / \partial u_i$, etc., will be denoted by $a_{(i)}, \alpha_{(i)}$, etc.

2. Consider now the total differential of any function $f(x; u_1, u_2, \dots, u_n)$ with respect to the u_i

$$(1) \quad df = f_{(1)} du_1 + f_{(2)} du_2 + \dots + f_{(n)} du_n.$$

Let this expression be squared and integrated with respect to x from 0 to 1. We obtain in this way the quadratic differential form in the variables $u_1, \dots, u_n$

$$(2) \quad \int_0^1 (df)^2 dx = \int_0^1 (f_{(1)} du_1 + f_{(2)} du_2 + \dots + f_{(n)} du_n)^2 dx \\ = \sum_{i,j=1}^n \left(\int_0^1 f_{(i)} f_{(j)} dx \right) du_i du_j = \sum_{i,j=1}^n E_{ij} du_i du_j,$$

where the expressions

$$(3) \quad E_{ij} = \int_0^1 f_{(i)} f_{(j)} dx$$

are functions of $u_1, u_2, \dots, u_n$.

3. If now we introduce a symbolic multiplication equivalent to ordinary multiplication followed by integration with respect to x from 0 to 1, we may write

$$(4) \quad \overline{df^2} = \overline{(f_{(1)} du_1 + f_{(2)} du_2 + \dots + f_{(n)} du_n)^2} \\ = \sum_{i,j=1}^n \overline{f_{(i)} f_{(j)}} du_i du_j = \sum_{i,j=1}^n E_{ij} du_i du_j,$$

where the bars indicate that the products are symbolic, i. e., integrated products. We have thus expressed the quadratic differential form

$$E = \sum_{i,j=1}^n E_{ij} du_i du_j$$

as the symbolic square of a linear form.*

4. It should be noticed that we have also

$$(5) \quad \int_0^1 [df(y)]^2 dy = \int_0^1 [df(x)]^2 dx = E$$

and again, if $\phi(x; u_1, u_2, \dots, u_n)$ is any function satisfying the conditions

$$(3) \quad \int_0^1 \phi_{(i)} \phi_{(j)} dx = E_{ij}$$

then

$$(6) \quad \int_0^1 [(d\phi(x))]^2 dx = E.$$

* Cf. Maschke, "A symbolic treatment of the theory of invariants of quadratic differential quantities of n variables," *Transactions*, vol. 4 (1904), pp. 445-469.

Such equivalent expressions for E may be employed in a manner analogous to the use of equivalent symbols in the symbolic theory.* In this paper the method indicated in equations (5) will be used for equivalent representation of E .

5. On account of the linear distributive character of integration, the ordinary function $f(x; u_1, u_2, \dots, u_n)$ satisfying conditions (3) may be taken as a *symbolic function* or *symbol* of the differential form E , and all the operations of the symbolic theory can be carried out if we agree that in products no more than two factors shall be expressed in terms of the same variable x ; that products of two factors written in terms of the same variable x shall be regarded as *integrated* products; and that a product of any number of factors expressed in terms of different variables $x, x_1, x_2, \dots, y, y_1, y_2, \dots$ shall be regarded as ordinary products.

We have then this result:

All of the identities of the symbolic differential invariant theory† may be interpreted as identities involving ordinary functions and their integrals.

6. The identities of the symbolic theory are given mainly in terms of determinants of the n th order. We denote the determinant

$$\left| \int_0^1 f_{(i)} f_{(j)} dx \right| \quad \text{by} \quad \frac{1}{\beta^2}.$$

The product of β and the Jacobian of n functions $a_1, a_2, \dots, a_n$, is denoted by

$$(a_1, a_2, \dots, a_n).$$

Using these notations we may write, for example, the symbolic expression

$(f_1, f_2, \dots, f_k, a_1, a_2, \dots, a_{n-k})(f_1, f_2, \dots, f_k, b_1, b_2, \dots, b_{n-k})$
(where $f_1, f_2, \dots, f_k$ are equivalent symbolic functions of the differential quantic E) in the form

$$\int_0^1 \int_0^1 [f(x_1), f(x_2), \dots, f(x_k), a_1, \dots, a_{n-k}] \\ \times [f(x_1), \dots, f(x_k), b_1, \dots, b_{n-k}] dx_1, \dots, dx_k,$$

where $\int_0^1 \int_0^1$ stands for $\int_0^1 \int_0^1 \dots k$ times.

* Maschke, loc. cit., p. 448.

† Clearly, the result holds equally well for the algebraic invariant theory.

7. Out of the large number of identities which could now be written down, we select the following for illustration.

$$\begin{aligned}
 & \int_0^1 \int_0^{(n-k)} [f(x_1), \dots, f(x_{n-k}), a_1, \dots, a_k] \\
 & \quad \times [f(x_1), \dots, f(x_{n-k}), b_1, \dots, b_k] dx_1, \dots, dx_{n-k} \\
 (7) \quad & = \frac{(n-k)!}{(n-1)!} \left| \int_0^1 \int_0^{(n-1)} [f(x_1), \dots, f(x_{n-1}), a_i] \right. \\
 & \quad \times [f(x_1), \dots, f(x_{n-1}), b_j] dx_1, \dots, dx_{n-1} \Big|, *
 \end{aligned}$$

where the determinant on the right is of the k th order, i and j running from 1 to k .

$$\begin{aligned}
 & \int_0^1 \int_0^{(k)} [f(x_1), \dots, f(x_k), f(x_{k+1}), \dots, f(x_n)] \\
 (8) \quad & \quad \times [f(x_1), \dots, f(x_k), a_1, \dots, a_{n-k}] \\
 & \quad \times [f(x_{k+1}), \dots, f(x_n), b_1, \dots, b_k] dx_1, \dots, dx_n \\
 & = k! (n-k)! (a_1, \dots, a_{n-k}, b_1, \dots, b_k). \dagger \\
 & \int_0^1 \int_0^{(n-k-r)} [f(x_1), \dots, f(x_k), f(x_{k+1}), \dots, f(x_{n-r}), b_1, \dots, b_r] \\
 & \quad \times [f(x_{k+1}), \dots, f(x_{n-r}), a_1, \dots, a_{k+r}] dx_{k+1}, \dots, dx_{n-r} \\
 (9) \quad & = \frac{(n-k-r)!}{(n-k)!(n-r)!} \sum_i \int_0^1 \int_0^{(n-r)} [f(x_1), \dots, f(x_{n-r}), a_i, \dots, a_{i,r}] \\
 & \quad \times [f(x_1), \dots, f(x_{n-r}), b_1, \dots, b_r] dx_{k+1}, \dots, dx_{n-r} \\
 & \quad \times \int_0^1 \int_0^{(n-r)} [f(y_1), \dots, f(y_{n-k}), a_{i_{k+r}}, \dots, a_{i_{k+r}}] \\
 & \quad \times [f(y_1), \dots, f(y_{n-k}), f(x_1), \dots, f(x_k)] dy_1, \dots, dy_{n-k}, \ddagger
 \end{aligned}$$

* Maschke, "Differential parameters of the first order," *Transactions*, vol. 7 (1906), p. 74, formula (8).

† Cf. Maschke, loc. cit., p. 453, formula (34) and p. 455, formula (61). Only the cases $k=1$ and $k=2$ are given by Maschke but this extension is easily proved.

‡ This formula is given in my paper "Vector interpretation of symbolic differential parameters," *Transactions*, vol. 11, p. 470.

the summation extending over all distinct combinations $i_1, \dots, i_r$ of r of the subscripts $1, 2, \dots, k+r$ and the order $i_1, i_2, \dots, i_{k+r}$ being so chosen that it may be obtained from $1, 2, \dots, k+r$ by an even number of transpositions.

8. We now consider a special case in which f is linear in the u_i and a set of n normed and mutually orthogonal functions $f_{(1)}, f_{(2)}, \dots, f_{(n)}$, of the variable x

$$(10) \quad f = u_1 f_{(1)} + u_2 f_{(2)} + \dots + u_n f_{(n)}.$$

We shall also suppose the functions a_i, b_i , etc., of the u 's to be linear, and write

$$(11) \quad a_i = \sum_{j=1}^n \alpha_{ij} u_j, \quad b_i = \sum_{j=1}^n \beta_{ij} u_j, \quad \text{etc.},$$

where the coefficients α_{ij}, β_{ij} , etc., are constant.

With every linear function a_i of the u 's we shall associate a function $a_i(x)$, defined by

$$(12) \quad a_i(x) = \sum_{j=1}^n \alpha_{ij} f_{(j)}.$$

We then introduce the notations

$$(13) \quad A_{(x_1, \dots, x_k)} = \frac{1}{k!} \begin{vmatrix} a_1(x_1), \dots, a_1(x_k) \\ a_2(x_1), \dots, a_2(x_k) \\ \vdots \\ a_k(x_1), \dots, a_k(x_k) \end{vmatrix}$$

$$(14) \quad F(x_1, \dots, x_n) = [f(x_1), f(x_2), \dots, f(x_n)].$$

We define the *complement* of A (written $\bar{A}$) by the formula

$$(15) \quad \begin{aligned} & (\bar{A})_{(x_1, \dots, x_{n-k})} \\ &= \int_0^{(k)} F(y_1, \dots, y_k, x_1, \dots, x_{n-k}) A_{(y_1, \dots, y_k)} dy_1, \dots, dy_k. \end{aligned}$$

The following formula may then be easily proved :

$$(16) \quad \begin{aligned} & k! (n-k)! A_{(x_1, \dots, x_k)} \\ &= \int_0^{(k)} [f(y_1), \dots, f(y_{n-k}), f(x_1), \dots, f(x_k)] \\ & \quad \times [f(y_1), \dots, f(y_{n-k}), a_1, \dots, a_k] dy_1, \dots, dy_{n-k}, \end{aligned}$$

$$(17) \quad A = \frac{1}{k! [(n-k)!]^2} \int_0^{(k)} F(y_1, \dots, y_k, x_1, \dots, x_{n-k})$$

$$\begin{aligned}
& \times \int_0^1 \binom{n-k}{i} \mathbf{F}(z_1, \dots, z_{n-k} y_1, \dots, y_k) \\
& \times [f(z_1), \dots, f(z_{n-k}), a_1, \dots, a_k] dz_1, \dots, dz_{n-k} dy_1, \dots, dy_k \\
& = \pm \frac{1}{(n-k)!} [f(x_1), \dots, f(x_{n-k}), a_1, \dots, a_k]. \\
(18) \quad \bar{\bar{\mathbf{A}}} &= \pm \mathbf{A}.
\end{aligned}$$

$$\begin{aligned}
(19) \quad & \int_0^1 \binom{k}{i} \mathbf{A}_{(x_1, \dots, x_k)} \mathbf{B}_{(x_1, \dots, x_k)} dx_1, \dots, dx_k \\
& \times \int_0^1 \binom{n-k}{i} (\bar{\mathbf{A}})_{(x_1, \dots, x_{n-k})} (\bar{\mathbf{B}})_{(x_1, \dots, x_{n-k})} dx_1, \dots, dx_{n-k}.
\end{aligned}$$

Substituting in formulas (7), (8), and (9), we have

$$(7') \quad \int_0^1 \binom{k}{i} \mathbf{A}_{(x_1, \dots, x_k)} \mathbf{B}_{(x_1, \dots, x_k)} dx_1, \dots, dx_k = \left| \int_0^1 \mathbf{a}_i \mathbf{b}_j dx \right|, *$$

$$\begin{aligned}
(8') \quad & \int_0^1 \binom{n-k}{i} \mathbf{A}_{(x_1, \dots, x_{n-k})} (\bar{\mathbf{B}})_{(x_1, \dots, x_{n-k})} dx_1, \dots, dx_{n-k} \\
& = (-1)^{k(n-k)} \int_0^1 \binom{k}{i} (\bar{\mathbf{A}})_{(x_1, \dots, x_k)} \mathbf{B}_{(x_1, \dots, x_k)} dx_1, \dots, dx_k,
\end{aligned}$$

$$\begin{aligned}
(9') \quad & \int_0^1 \binom{j}{i} (\bar{\mathbf{A}})_{(y_1, \dots, y_j)} (\bar{\mathbf{B}})_{(x_1, \dots, x_k y_1, \dots, y_j)} dy_1, \dots, dy_j \\
& = \frac{k! j!}{(k+j)!} \sum_i \mathbf{A}_i(x_1, \dots, x_k) \int_0^1 \binom{j}{i} (\bar{\mathbf{A}})_i(y_1, \dots, y_j) (\bar{\mathbf{B}})_{(y_1, \dots, y_j)} dy_1, \dots, dy_j,
\end{aligned}$$

where $k+j=n-r$ and where i has the same meaning as in formula (9).

UNIVERSITY OF MISSOURI,
July, 1910.

* This identity is proved directly by Professor R. G. D. Richardson and Dr. W. A. Hurwitz in the paper "Note on determinants whose terms are certain integrals," BULLETIN, vol. 16, p. 16.

POINCARÉ'S GÖTTINGEN LECTURES.

Sechs Vorträge über ausgewählte Gegenstände aus der reinen Mathematik und mathematischen Physik. Von HENRI POINCARÉ. Leipzig and Berlin, Teubner, 1910. 8vo. 60 pp.

IN these six Göttingen lectures, delivered April 22–28, 1909; at the invitation of the Wolskehl commission, Poincaré has treated a wide range of interesting subjects in a masterly and illuminating way. It is in the nature of the case that the methods and results should be outlined with exceeding brevity, and this makes the little book hard reading. Fortunately, the reader who is not content with the summary discussions given in it can supplement them for the most part by the use of recent articles by Poincaré.* The topics in their order are (1) the Fredholm equations, (2) the application of the theory of integral equations to fluid motion, (3) the application of the theory of integral equations to Hertzian waves, (4) the reduction of Abelian integrals and the theory of Fuchsian functions, (5) transfinite numbers, (6) the new mechanics. The sixth lecture was popular in its nature and was delivered in the French language.

1. The Fredholm equations. The integral equation of the second kind

$$\phi(x) = \lambda \int_a^b f(x, y)\phi(y)dy + \psi(x)$$

is known to admit two formal solutions, namely the Neumann solution as a power series in positive integral powers of the parameter λ , which converges for small values of λ , and the Fredholm solution as the quotient of two entire functions of λ . Poincaré first derives the fundamental formula for $\log D(\lambda)$, where $D(\lambda)$ is the denominator of the Fredholm resolvent, by a count of combinations, and then defines the numerator at once by a use of the Neumann formula for the resolvent. By this method of comparison a clear analysis of the solution of the integral equation is obtainable. A natural extension of the method enables Poincaré to treat the important case where the

* First lecture: *Acta Mathematica*, vol. 33 (1909), pp. 57–86. Third lecture: *Palermo Rendiconti*, vol. 30 (1910), pp. 169–259. Fourth lecture: *Palermo Rendiconti*, vol. 29 (1909), pp. 281–336. Fifth lecture: *Acta Mathematica*, vol. 32 (1908), pp. 195–200, and *Revue de Métaphysique et de Morale*, 1909, pp. 461–482.

kernel $f(x, y)$ becomes infinite but some iterated kernel $f_n(x, y)$ remains finite. Fredholm has shown that a solution exists; but in his formulas there remained a common factor in numerator and denominator, which is here removed by the use of a modified resolvent. This resolvent is obtained in a very simple way by striking out certain terms from the Fredholm resolvent.

Some partial results for the case when $f(x, y)$ and all of its iterated kernels become infinite are stated, and the lecture concludes with a consideration of two special integral equations of the first kind, reducible to Fredholm equations by means of Fourier's integral and series.

2. Application of the theory of integral equations to fluid motion. In this lecture and the succeeding one are considered typical and important examples of the way in which problems of mathematical physics lead to Fredholm equations.

The first problem is to determine the fluid motion in a sea of varying depth on a rotating earth under the influences of periodic perturbative forces. If the attractive forces due to water displacement be neglected, there results a non-homogeneous linear partial differential equation of the second order in two independent variables. It is this equation which is considered. When a vertical wall surrounds the sea the method of Hilbert and Picard is applied to construct the kernel or Green's function of the associated integral equation, which is of the first kind. The equation can then be treated by a method due to Kellogg, or by a method of integration in the complex plane given by Poincaré, which replaces the given integral equation by an equivalent one of the second kind. If the shore of the sea is not vertical, the corresponding line is a singular line of the differential equation. Also a second line (critical latitude) is now taken into account. A similar reduction can be again effected by a succession of three steps. In this way a solution is proved to exist in both cases. Finally it is indicated that no new difficulty appears if one does not neglect the attraction due to water displacement.

3. Application of the theory of integral equations to Hertzian waves. Poincaré investigates the phenomenon of the curvilinear propagation of Hertzian waves on the earth's surface, which accounts for the great distances over which wireless telegraph messages may be sent. This phenomenon has been attributed to the great length of the Hertzian wave as compared with that of the light wave, which is propagated in a straight line. A quanti-

tative mathematical discussion is here given. By regarding the earth as the outer conductor and the transmitter as the inner conductor, and by considering damped synchronous vibrations, one obtains an integral equation of the second kind to determine the electrical density μ produced on the earth's surface. But this merely yields an existence theorem. The lecturer brings the problem to a practical conclusion by obtaining an approximate expression for μ ; the method depends on an expansion of μ in Legendre polynomials, and the use of asymptotic formulas. It turns out that the curvature increases with an increase in the length of the wave, and with a decrease in the distance of the transmitter from the earth.*

4. Reduction of Abelian integrals and the theory of Fuchsian functions. If a system S of Abelian functions is given which is reducible in terms of a second system S' , and if both S and S' arise from algebraic curves C and C' , there will be an algebraic correspondence set up between groups of points on the curves; only that case is considered in which to one point of C corresponds one point of C' , while to one point of C' correspond n points of C . There then exist certain reducible integrals which belong to C , and the table of periods has a simple normal form. Also the number n is equal to the order of a related theta function, as is proved. Now it is known that there exists a Fuchsian function which uniformizes any algebraic curve, in particular C' . The fundamental polygons for the curve C' may be taken to be limited by arcs of circles, and each such polygon for C to be formed by n of these polygons for C' . The interplay between the integrals and the polygons gives rise to numerous geometric facts about the curves C and C' , and about congruent polygons in space of constant negative curvature. Several examples are given which illustrate the beautiful theorems concerning C and C' that may be thus obtained.

5. Transfinite numbers. The lecture develops Poincaré's attitude toward some of the subtleties in this controversial field of mathematics. The main ideas for which he contends are two in number: first that no mathematical entity exists that is not definable in a finite number of words, and secondly that all definitions must be what he calls 'predicative.' For example, Poincaré objects to the familiar proof that every algebraic equation $f(x) = 0$ has a root, dependent on the exis-

* It is noted at the end of the lecture that the final conclusions will need modification owing to the fact that important terms have been overlooked.

tence of a minimum of $|f(x)|$. For it would not be possible from his standpoint to speak of the totality of values of $f(x)$ without meaning to refer to those values for which x is defined in a finite number of words, and this is not permissible since the notion of the totality of definable values of x is non-predicative. The difficulty involved is that there are elements of this 'totality' which themselves are defined in terms of the 'totality,' and hence the notion embodies a vicious circle. The elucidation of the meaning of the word 'predicative' given in the lecture is not clear.

Poincaré begins by considering the apparent contradiction between Richard's proof (based on the first of the above-stated ideas) that the continuum is denumerable, and Cantor's proof that it is not denumerable, but it is shown that this contradiction is not real, since Richard employs a non-predicative definition.

He then passes on to point out how the demonstration of the theorem that every algebraic equation $f(x) = 0$ has a root needs to be stated from his point of view.

Several other matters are briefly touched upon in conclusion. The Bernstein theorem is correct for Poincaré, and the problem of the well-ordering of the continuum (in Cantor's sense) seems meaningless to him. Moreover, he is not convinced that the second transfinite cardinal number exists. These conclusions are all in thorough-going harmony with the two main ideas.

The principal objection that may be urged against these views of Poincaré's (and of many other mathematicians) is the practical one that they so closely restrict the notion of a *class*. However it is the intuition rather than the logical faculty that rebels against their acceptance. Judging by the past, this fact is an indication in favor of the views presented; they form one step toward the elimination of the infinite from mathematics, and one may well doubt whether the conception of an infinite class as having objective existence will play any ultimate rôle in rigorous mathematics.

6. The new mechanics. In this closing popular lecture, the modifications that mechanics may have to undergo as a result of recent advances in physics are considered. If the final equations of motion are those of the electromagnetic field, and experiments seem to indicate this, startling conclusions follow: the velocity of all bodies is less than that of light and their inertia increases with their velocity; it will be

impossible to tell by any experiment whether one is at rest or in uniform motion of translation with respect to the ether; one cannot say that two events are simultaneous in an absolute sense; moreover, all bodies will undergo a shortening in the direction of their motion. It is with this fascinating subject, more especially with the modified mechanics that it implies, that Poincaré deals.

It is only in bodies that possess a very great velocity that one can hope to discern a deviation from the laws of the Newtonian mechanics. Now Mercury moves at the greatest velocity of any of the planets and it is precisely Mercury that possesses a small anomaly not yet explained. The new mechanics accounts for a part of this, as Lorentz has shown, and nowhere else produces a sensible modification in the motion of the planets.* After these facts have been presented, Poincaré concludes by observing that the Newtonian mechanics will remain forever the mechanics at velocities which are small with respect to the velocity of light, and thus will continue to preserve its fundamental importance.

G. D. BIRKHOFF.

THE THEORY OF ELECTRONS.

The Theory of Electrons and its Applications to the Phenomena of Light and Radiant Heat. By H. A. LORENTZ. Leipzig, B. G. Teubner, 1909. 332 pp.

Ueber Elektronen. Von W. WIEN. Zweite Auflage. Leipzig, B. G. Teubner, 1909. 39 pp.

THE physical hypotheses which are adopted for the formulation and discussion of the correlation of a certain restricted group of physical phenomena, and in particular for the development of their mathematical theory, depend to a large degree upon those phenomena themselves and are in no small measure independent of other groups of related phenomena and of the point of view adopted relative to physics as a whole. Thus in dealing with most problems in heat, the old idea of heat as a sort of massless substance, caloric, still serves as the simplest

* Newcomb and Seeliger have shown that the anomaly in the motion of Mercury and of the other inner planets can be explained, for the most part, by the attraction of matter, diffusely distributed about the sun, of plausible mass, and of such distribution as to give rise to the known phenomenon of the zodiacal light.

hypothesis, and is and should be used despite the facts that for some phenomena in heat the kinetic theory is necessary and that for the proper point of view relative to physics as a whole heat is regarded as a mode of motion. A similar, though more highly diversified, state of affairs exists in regard to electricity. There is still ample field for the application of the old hypothesis of one or two electric fluids, there is still occasion to avail oneself of the hypothesis of action at a distance as well as of that of action through a medium, and furthermore there are now coming more and more into the foreground some phenomena which are best treated by adopting the hypothesis of atomic electricity—at any rate as far as negative electricity is concerned. It is with this latest hypothesis, this atomistic view of electricity that Lorentz's Theory of Electrons deals.

The application of the atomic hypothesis to electricity is by no means of very recent date. It was foreshadowed already in Faraday's work on electrochemical equivalents; it was stated with clearness and precision by Helmholtz in his Faraday Lecture nearly thirty years ago; it has for a long time been the subject of numerous theoretical and experimental researches on the part of many physicists of whom J. J. Thomson and Lorentz are perhaps the most eminent. In the last fifteen years the development and application of the atomic view has reached such proportions as to constitute of itself a vast science. Some years ago the publication by H. Abraham and P. Langevin of their large collection of memoirs by various authors on the general subject of ions, electrons, and corpuscles * did much to render the literature easily accessible; but, as a connected, theoretical, and detailed account of electrons and their applications has for some time been sadly needed, it is particularly fortunate that we have now just such a work from the pen of one who is himself the founder and builder of so great a part of the whole theory. The lectures which formed the foundation of the book under review were delivered in this country at Columbia University in 1906. The years between the delivery of the lectures and their publication saw many advances in the atomistic theory of electricity and of these advances those which are most germane to the point of view of the lectures have been incorporated in the volume.

* *Les Quantités élémentaires d'Électricité—Ions, Électrons, Corpuscules—mémoires réunis et publiés par H. Abraham et P. Langevin, Paris, Gauthier-Villars, 1905. xvi + 1136 pp.*

Chapter I deals with general principles and the theory of free electrons. The author adopts the same system of units and the same notation as in his articles in the fourth volume of the *Encyclopædia*; he uses vector analysis. It is interesting to notice the attitude he takes toward vector analysis—about the same attitude as he takes toward the choice of units—one of convenience. Nearly two pages are required to construct his system of vector analysis. The next thing to do is to set up the equations of the free ether. Then, after mentioning the possibility of deriving the equations for ponderable bodies (at rest) by the introduction of certain constants, the author settles down to showing the necessity of the electronic or atomistic point of view and to setting up the desired equations for the electrons. The integration of the equations by means of the scalar and vector retarded potentials is given. With this groundwork everything is ready for a rapid development of such vital points in electromagnetic theory as the Poynting vector and the flow of energy, stresses in the medium, the pressure of radiation, electromagnetic momentum, (approximately) uniform motion of a charge, electromagnetic mass, the vibration of a charge and the radiation of energy, the electromagnetic equations referred to moving axes, and the kinetic theories of Drude. The arrangement of the text is noteworthy. The details of mathematical analysis are relegated to notes at the end of the book (where they occupy 100 pages) and thus the main text is kept free for the presentation and discussion of the results. The same method is pursued in the later chapters. There is nothing dogmatic about the handling of the material, the author discusses his problems frankly and with every attempt to explain the why as well as the how.

It is but natural that the first chapter should be rather plain sailing and that its chief interest should lie in the artistic presentation of the necessary introductory material. With the second chapter, on emission and absorption of heat, we come to the consideration of some of the most troublesome questions of modern physics, lying in the widely different fields of kinetic theory, thermodynamics, and electromagnetism. The author treats Kirchhoff's law, the Boltzmann-Stefan law, and Wien's displacement law. He then turns his attention toward the further determination of the law of distribution of energy in the spectrum of a black body. Here three different theories are presented and discussed—that of Planck, which leads to

the most complete determination of the function but is not treated in detail because it is hardly germane to the method of this book, that of the author himself, which is a beautiful combination of kinetic and electric theories but is valuable only for large wave lengths, and that of Jeans, which is more dynamic in nature but does not admit a strict state of equilibrium. The discussion and comparison of these theories is clear and illuminating. Despite the excellencies of the theories of Jeans and of the author, our own inclinations are strongly toward that of Planck; its foundation upon the rather vague mechanism of the resonator may be hardly electromagnetic, its use of the theory of probability may be scarcely canonical, its implied quasi-atomic structure of energy may be distinctly annoying, but somehow it does get very close to the facts and in time its difficulties may evaporate.*

Chapter III on the theory of the Zeeman effect is a very pretty piece of work. After discussing the simplest case of the Zeeman effect the question of series of spectral lines is treated. There then follows an examination of the general properties of a vibrating system of electrons and of the special properties of certain simple systems. The difficulties of a complete explanation of the phenomena involved are thoroughly presented. The author is bent on stating the importance of the problem and the need of its solution much more than on over-emphasizing the value of work already accomplished; he even concludes that "after all, we are rather at a loss as to the explanation of the complicated forms of the Zeeman effect." There can be little doubt, however, that he is on the right track and has contributed largely to the ultimate resolution of the difficulties of the analysis of the spectrum. The very diversified spectra of different elements, the astonishing simplicity of some and the bewildering complexity of others, will occupy the attention and perhaps thwart the endeavors of mathematical physicists for a long time to come.

Intimately connected with the foregoing and an immediate extension of it is the work of Chapter IV on the propagation of light in a body composed of molecules and the theory of the

*It may be remarked that Larmor has recently given an exposition of Planck's theory which apparently removes the difficulty as to the structure of energy. Another most remarkable fact is that, although Planck by his method found a value for e (4.7×10^{-10}) about fifty per cent. larger than that which had long been accepted, some recent investigations seem to corroborate his theoretical value. A showing like that means a great deal.

inverse Zeeman effect. The author passes from his equations for electrons to those for ponderable bodies by a system of averages executed over large numbers of electrons and he is able to give the theory of dispersion of light with the particular hypothesis that the electrons, which are now so well known to be fundamental, are those small sensitive particles which occur with no specializing hypothesis in the theories of Sellmeyer and others. Although the author makes no mention of Gibbs, it may be well to recall that Gibbs, in a series of papers, remarkable for generality rather than specialization of hypothesis and culminating in his deduction of the general equations of monochromatic light in media of every degree of transparency, constructed an electromagnetic theory of optics on a foundation of averages. In a problem like this it is decidedly advantageous to distinguish those results which may be obtained by general consideration from those which follow from special hypotheses. The author's work with electrons leads to a number of formulas which may be compared with experimental evidence to show their accuracy and to emphasize the value and essential truth of the physical model which he uses. After touching upon the relation of Faraday's rotation of the plane of polarization to the Zeeman effect and upon some of Voigt's results in magneto-optics, the author again closes with his characteristically modest warning that the theories he has set forth can hardly be regarded as more than a start toward the final solution of the problems. A perusal of the chapter will, however, convince the reader that he is reading some very real up-to-date vital physics.

The last chapter is devoted to optical phenomena in moving bodies. Here again, as constantly throughout the volume, we are in the presence of theories which are essentially due to the initiative of Lorentz. Here again there is no haste to assume any particular hypothesis without due consideration; the points of view of Stokes and Fresnel are examined before passing to the strict theory of electrons. The experiments of Michelson and Morley and of Rayleigh and Brace are discussed. The significance of the transformations of what Poincaré has called the Lorentz group is made clear. Some remarks relative to the rigid electron of Abraham and the electron of constant volume of Bucherer and Langevin are inserted. Finally the work closes with a brief reference to the principle of relativity as formulated by Einstein. At present the principle of relativity,

owing to pioneer work of Einstein and Minkowski and the enthusiastic cultivation of their followers, is receiving a great deal of attention. Lorentz speaks somewhat regretfully of the fact that he cannot give it greater space in his book. But we must remember that the methods employed by Einstein, Minkowski, and others in developing this principle are Maxwellian rather than Lorentzian, that they are not essentially concerned with electrons, and that consequently they are not really a proper part of the present work.

In fact, what some students of relativity would do with electrons, especially with vibrating electrons emitting trains of waves, is hard to imagine; for there are not a few who seem intent on abolishing the ether and on disregarding the phenomena which were the prime cause for its introduction. We may all come to this — one never can tell. If Lorentz's ether-electron model of the universe is so highly valuable as a basis for some domains of physics, there may well be other nearby domains for which the etherless kinematic space of some of the relativists is equally useful. And it will not do to overlook J. J. Thomson's Faraday-tube model! To the mathematical purist it may seem frivolous to take such a broad, smiling attitude toward the coexistence of such different models in a single scientific field; that is because the pure mathematician is an esthesiogenist rather than a scientist. So careless is the real scientist in the presence of divers models in the same field that we may safely venture the guess that even among specialists on heat, thermodynamics, and kinetic theory there are few who from the kinetic point of view can readily establish the equations for the flow of caloric in a marble column; they have more important and interesting things to do.

To pass from Lorentz's work to the little essay of Wien is but a step. It was in 1906, we believe, that Wien delivered an address before the *Versammlung deutscher Naturforscher und Aertze* on electrons. The address was printed, and has now appeared in a second edition to which some material of recent date has been added. As the pamphlet is itself a very succinct review of a wide field, it would be futile to review it here further than to mention its popular and yet scientific nature which must recommend it to all who are interested in what is going on in live physics and who have not yet acquired Wien's broad view over the field of electron theory. For those

really unfamiliar with the subject it would be well to read Wien before Lorentz and then again afterward. It is an excursion that is worth while for anyone. Come, drop an epsilon just for once and pick up an electron; it is a deal larger, even if its radius is only 10^{-13} cm.

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BOSTON, MASS., August, 1910.

SHORTER NOTICES.

Vorlesungen über Differentialgeometrie. By R. v. LILIENTHAL.
Erster Band: *Kurventheorie.* Leipzig, B. G. Teubner, 1908.
8vo. vi + 368 pp. 12 Marks.

It is undoubtedly one of the achievements of the mathematical development of recent years to have shown that mathematical rigor is not the same as pedantry; that a theory may be presented in an irreproachable manner and still be attractive and interesting. There can be little doubt from a pedagogic point of view that if, in a treatise intended for students, a choice must be made between a comprehensive but tedious, and a less comprehensive but interesting treatment, the latter should be given preference, provided of course that its limitations be properly indicated. Both of these general principles seem to have been ignored completely in the construction of the book under review.

The author wishes to be rigorous and general. For this purpose he excludes all considerations involving infinitesimals. One might begin to quarrel with him on this score, since a properly formulated notion of infinitesimals is not altogether unknown. He then confines his attention exclusively to the case in which all of the functions which occur are analytic, a restriction of generality which seems extraordinary from the author's point of view, since all of his developments require only the existence of a finite number of derivatives. But even then, also for the sake of rigor, he excludes (as a matter of course, he says), the consideration of questions of order of contact, although all of these questions are easily treated by power series methods and would fall most naturally into the theory as developed by him.

It is true, historically, that differential geometry has up to the present time been occupied primarily with metric questions. We should doubtless also exercise a certain amount of patience with those who are accustomed to think of differential geometry as exclusively metric, and who refuse to bother with projective considerations in their own work. Still, in a comprehensive treatise on differential geometry, either the projective theory should be included, or else the exclusion should be explicitly indicated by the introduction of some qualifying adjective into the title; we should suggest *metric* differential geometry. The author does, however, mention briefly one phase of projective differential geometry by discussing the curves which belong to a linear complex. The following amusing remark is of interest, however, in showing how far removed he is from the modern spirit of geometry, in which the change from one space element to another has turned out to be so exceedingly fruitful. He says: "Die Plücker'sche Theorie ist entweder ein Beispiel dafür, dass rein formale Überlegungen zu wichtigen geometrischen Begriffsbildungen führen können, oder sie ist, erst nachdem sich ihr Urheber auf anderem Wege von der Bedeutung des Komplexbegriffes überzeugt hat, unter dem Vorbilde der gewöhnlichen Theorie der Ebene und der Flächen zweiten Grades aufgestellt." He obviously means, as the context shows, that the importance of the notion of a complex consists entirely in its applicability to kinematics.

Another omission which, to the reviewer, seems a serious one, is that no mention is made of intrinsic geometry as developed principally by Cesàro. The general theorems which are connected with this point of view, and the added strength which it gives to the investigator, are so valuable that no treatment of the theory of curves which goes beyond the elements can be considered adequate which does not at least give some account of this theory.

In spite of these criticisms, there are a number of things in the book which deserve high commendation. The treatment of one-parameter families of plane curves is very careful, and involves some new points of view, although the unusual distinction between envelope (Einhüllende) and curve of contact (Berührende) is of doubtful value.

The ordinary treatment of the differential geometry of plane curves is essentially a geometric interpretation of the first and second derivatives by means of the tangent and the circle of

curvature. A geometric interpretation of the third derivative is due to Abel. Transon (1841) who introduced the notions aberrancy of a curve, and axis of aberrancy. This notion has been almost completely lost, and the author is to be commended for reviving it. Transon's term, however, was "axis of deviation," which just reverses the historical statement as given by the author in regard to these two names.

The book is carefully printed and none of the misprints noticed by the reviewer can give rise to any difficulty.

E. J. WILCZYNSKI.

Elliptische Funktionen. Von Professor Dr. KARL BOEHM. Erster Teil. Götschen (Sammlung Schubert XXX). Leipzig, 1909. xii + 354 pp. 8.60 Marks.

A TREATISE on any of the functions of analysis, the properties of which are well known, must rely for its usefulness upon the mode and style of presentation. The historical development may be followed, or the functions may be introduced through later discovered properties. There can be no doubt that the former is the natural and more easily comprehended introduction, especially to the higher functions. Professor Boehm has elected the latter course, in this first volume, with the understanding, however, that the reader may commence with the second volume which starts out with elliptic integrals and the inversion problem.

The present volume is occupied with the various infinite series which represent simply and doubly periodic functions, with related series and products, and with their mutual interdependence.

The beginner will probably do well to take the author's suggestion and commence with the second volume. Students who have had a good course in the calculus can easily appreciate the inversion problem and its close proximity to so-called applications, but would most likely become discouraged and cry *cui bono* if requested to assimilate the contents of this volume in order to become acquainted with elliptic functions. This is true even if the shorter course were followed which the author has carefully planned and indicated by footnotes at the proper places through the volume.

It must be said, however, that the volume contains all the necessary preparation for an understanding of the series repre-

senting the elliptic functions. Indeed we do not meet with these series until we have read nearly one-third the volume, or through the first four chapters. These introductory chapters are occupied with infinite series and infinite products in general and with the particular series and products which represent simply periodic functions — the latter for the purpose of analogy. The author takes advantage of these early pages to introduce ideas and a terminology peculiar to the theory of functions in general and to periodic functions in particular. Thus the reader learns about zeros and poles, periodicity and homogeneity, even and odd functions, polarized functions (after Méray); holomorphic and meromorphic functions, etc.

We begin the study of the series leading to the elliptic functions at the fifth chapter. The author avoids the Mittag-Leffler theorem (it is mentioned with a reference to the *Acta Mathematica* on page 34) and considers de novo the convergence of the series

$$\sum_{n=-\infty}^{n=+\infty} \sum_{m=-\infty}^{m=+\infty} [n + m\pi + n\theta]^{-h}.$$

This is also done for the analogous simply infinite series in the introductory chapters. The work seems unnecessarily long and tedious in consequence, but a deeper insight into the nature of the convergence is perhaps thereby gained. For example, in section 46 the above series for $h = 2$ is considered between the *finite* limits $-n_1, +n_1; -m_1, +m_1$. The limit toward which the series converges for infinitely increasing values of m_1 and n_1 is shown to depend upon the limit of the ratio $m_1 : n_1$ and is finite only when the latter limit is finite. It is doubtful, however, if the knowledge gained is worth the effort to master the longer, if more elementary, processes entailed as well as the sacrifice of a definite statement and proof of the Mittag-Leffler theorem as a point of departure. At the end of the chapter, which is forty pages in length, we are in possession of the conditions for convergency of the basal series for $h = 1, 2, 3$, together with some of the properties of the functions to which these series lead; i. e., $\zeta u, pu, pu$.

Chapter six is given up to the consideration of the sigma function. The same general method is followed, beginning with the consideration of the convergency of the proper doubly-infinite product and ending with the development in trigonometric functions.

After an excellent chapter on congruent complex numbers with respect to a double modulus, we come in chapter eight to the first explicit definition of an elliptic function, viz., an elliptic function is a doubly periodic function which is meromorphic over the finite region. Then follow the Liouville theorems concerning the sum of the residues, the equality in number of zeros and poles, and the congruency of the sum of zeros to the sum of the poles in a period parallelogram.

We have now completed the groundwork of the theory and it remains to fill in the details. Some points in the arrangement of this detail may be noted. The development of the primary Weierstrassian functions in power series is put off to the eleventh chapter after a long discussion (chapter ten) devoted to the less readily comprehended Hermitian elliptic functions of the second and third kinds. Again the Jacobi theta functions, the sigmas with index, and the Gudermarian snu , cnu , dnu , are not introduced until the twelfth and last chapter.

One might naturally hesitate to adopt this arrangement in presenting the subject to students for the first time, even if the above series were elected as the point of departure. A more or less ready familiarity with the primary functions and the modes of reducing them to numerical computation would seem advisable before proceeding to the more complex functions which can be expressed in terms of them.

On the other hand, it must be said that there is a certain gain in elegance when all the elliptic functions of first, second, and third kinds are introduced by means of their formal developments in infinite series and the property of periodicity together with a proof that they can all be expressed in terms of a relatively small number of primary functions. The detailed discussion of these primary functions would then follow and complete the course. That Professor Boehm has, in the main, adopted this latter arrangement of material makes his book appeal more to the advanced reader than to the beginner.

The book is consistent throughout in adhering to the main line of thought above sketched and thus avoids confusing the reader by presenting a number of points of departure with their consequent developments. While one may not agree with the author as to arrangement of material, one must admire the style and general effect of the book. It is interesting and valuable to have consistent and parallel developments of the simply and doubly periodic functions brought together in a convenient form.

L. WAYLAND DOWLING.

Sammlung von Aufgaben zur Anwendung der Differential- und Integralrechnung. Von FRIEDRICH DINGELDEY. Erster Teil: *Differentialrechnung.* Leipzig and Berlin, Teubner, 1910. (Sammlung Teubner, Band XXXII, 1). vi+202 pp.

THIS volume, the first of two covering the field of the calculus, contains a collection of 580 exercises illustrating the application of the differential calculus to the problems of geometry, physics, and engineering. The author is warranted in making the assumption that in the collections made heretofore the problems in the three fields have been too much isolated; and, working with the conviction that they should be united in one volume, he has arranged in the present book a most excellent combination of such geometric problems as might interest the physicist and engineer, together with physical and technical applications of the differential calculus which ought to appeal to the students of pure mathematics.

The book contains not only the exercises to be solved, but also all necessary explanations, indicated steps in the solutions of some of the problems, and answers to all. At the beginning of each chapter is given a statement of the principles of the calculus to be used, which is more than a collection of rules to be applied arbitrarily, and which falls short of a treatise on the theory. The specific purpose of the book is to supplement, not to furnish, the theory of the calculus. Numerous references to mathematical journals and published texts should inspire a further investigation on the part of the student into the problems proposed. Since it is intended that the book be used in connection with a first course in the calculus presented by lectures, there is necessarily quite a list of drill exercises included in the twenty chapters, covering the usual subjects discussed in a thorough first course in the differential calculus. However, the problems, for the most part, are of a type above furnishing mere drill in applying principles and possess mathematical interest of their own, while many may be called classics in the study of the calculus. These are well suited for, and could be used to great advantage in connection with a second course in that subject. A systematic study of the various steps in the solutions of these exercises would be of the greatest benefit to the student, and would exhibit clearly the many fields of mathematics entered in the study of the calculus. A glance at the index reveals such terms as absolute convergence, Bernoulli's numbers, Wronskian and functional determinants,

Hessian curves, least squares, line coordinates, potential, and the like, all of which when used are accompanied by explanations of their nature sufficient for the purpose of the problems.

While the author in the preface would give the impression that the fields of geometry, physics, and engineering are equally represented in the collection, yet even a casual glance reveals that the geometric side of the calculus is emphasized more than the other two, and an abundance of interesting geometric properties of curves and surfaces is brought out. Problems involving numerical calculations and the use of tables are brought in frequently, a procedure calling for definite results, and well worth while.

Instructors of the calculus will find in the collection much material to supplement any course given where new problems are welcome; while authors of text-books on the subject might easily receive many valuable suggestions from a perusal of its pages.

The volume is in every way typographically, both in print and figures, up to the high standard of the *Sammlung Teubner*; in its specific field it fills a practical need, and because of the excellence of the first part the volume on the integral calculus is eagerly awaited.

ERNEST W. PONZER.

Differential and Integral Calculus. By PROFESSOR DANIEL A. MURRAY. New York, Longmans, Green and Company, 1908. xviii + 491 pp.

THE aim of the author of this text-book has been (to quote from the preface) "to describe and emphasize the fundamental principles of the subject in such a way that, as much as may reasonably be expected, they may be clearly understood, firmly grasped, and intelligently applied by young students"; and again, "the aim has been to write a book that will be found helpful by those who begin the study of calculus without the guidance and aid of a teacher." This is by no means a simple undertaking in view of the inherent and essential difficulties of the subject. The notion of a limit is fundamental; and while students readily acquire more or less vague ideas on this subject, it seems to be difficult for most of them to get a clear and accurate conception of it. This difficulty must be squarely faced; fundamental definitions and principles must be set forth in language that is accurate and therefore necessarily technical.

Any attempt to state, restate, or explain these principles in language that is less technical, and therefore inaccurate or even meaningless, is not conducive to real clearness. The beginner in calculus is not helped by having the inherent difficulties of the fundamental concepts explained away. And into this sort of error, the author of the book under consideration would seem, in the judgment of the reviewer, to have fallen to some extent.

A few specific cases may be cited in support of this opinion. In the very beginning, the limit idea cannot be avoided in defining the derivative, for the reason that the fraction $\Delta y/\Delta x$ is *meaningless* when $\Delta x = 0$. When one writes, for example,

$$y = x^2, \quad \Delta y = 2x\Delta x + \Delta x^2, \quad \frac{\Delta y}{\Delta x} = 2x + \Delta x,$$

the third equation is true *except when* $\Delta x = 0$. The author seems to make no distinction between the two members of this equation. Thus, on page 6, we read, " $\Delta y/\Delta x$ will reach the value 4 when Δx decreases to zero," but $\Delta y/\Delta x$ has not been defined to mean anything other than Δy divided by Δx . On page 9 one finds, "one can confidently and accurately state what these ratios [$\Delta s_1/\Delta t_1$ and $\Delta y_1/\Delta x_1$] *will become* [the italics are from the book] when Δt_1 but Δx_1 actually reach zero." This latter case is hardly excused by a note immediately following which says, "Moreover, it should be carefully noted that *at the final stages* in the solutions of the problems in Arts. 3 and 4, $\Delta s_1/\Delta t_1$ is *not* regarded as a fraction composed of *two* quantities, Δs_1 and Δt_1 , but as a *single* quantity, namely the *speed* after t_1 seconds." The symbol was certainly introduced into the problem as a fraction. Just when does it cease to be a fraction?

On page 29 we read, "It is evident that the reciprocal of an infinitesimal approaches a number which is greater than any number that can be named, namely, an infinite number. Accordingly, an infinite number may be defined as the reciprocal of an infinitesimal." Does this mean anything? It may be allowable to use the notation $x \doteq \infty$ *after one has explained exactly what such notation is to mean*; but it is doubtful whether there is ever justification for writing (as the author does, page 185) $F'(x) = \infty$, or for speaking of "the equation, $\phi'(x) = \infty$ " (page 117). An excellent definition of an asymptote as the limiting position of a tangent is given on page 200; but later,

on page 202, we read that a certain line "is tangent to the curve at the infinitely distant point."

Other examples of rather loose reasoning are noticeable. On page 47, having proven that

$$\frac{d}{dx} [f(x) + c] = \frac{d}{dx} [f(x)],$$

the author says, "It follows from (1) [the equation just cited] that the derivative of a constant is zero"; but the distributive law has not yet been proven. In a note at the bottom of page 117 we read, "It follows . . . that if $\phi(a)$ is a maximum (or minimum) value of $\phi(x)$, then . . . $\sqrt{\phi(a)}$ is a maximum (or minimum) value of $\sqrt{\phi(x)}$," which is obviously not true for $\phi(x) = x^2$. The argument on page 275 holds only for a continuously increasing function, which restriction is not stated in the text. The statements (page 200) that "an ellipse cannot have an asymptote" and "the parabola $y^2 = 4px$ has no asymptote" are flat contradictions of the statement (page 203) that "a curve whose equation is of the n th degree has n asymptotes, real or imaginary." But the reader is still more likely to be confused when he finds (page 163) an unqualified "proof" of a property of the evolute followed by a note saying, "Property (b) should not be applied thoughtlessly; for in certain circumstances, for either the curve or its evolute, the property does not hold." An instance is then cited where the property "does not hold"; but no explanation is given of this paradox of having proven something which is not true. The note is concluded by the phrase, "which is obviously absurd"; and this is certainly true in a sense probably not intended.

One's estimate of the text-book will depend entirely on whether he regards such errors as are cited above as serious and fundamental or merely trivial—if, indeed, he regards them as errors at all. The book certainly has many commendable features.

There are numerous references throughout, both to other texts of the same grade and to books where the subjects are treated more fully and rigorously. These will undoubtedly prove helpful to student and teacher alike.

Taylor's theorem is derived by extending the mean value theorem; thus giving emphasis to the remainder from the beginning, which is most desirable. Newton's application of

the mean value theorem to the finding of approximate roots of equations is given, and this feature should prove to be of practical value, especially when applied to transcendental equations. The examples given are unfortunately largely rational algebraic equations, for which Horner's method is probably simpler.

The chapters relating to space geometry and kinematics are interesting and attractive. A nice distinction is made between speed and velocity, leading to the notion of normal as well as tangential components of acceleration.

The part devoted to integral calculus includes a chapter on center of mass and moment of inertia. This is followed by a chapter containing a brief treatment of ordinary differential equations of the first and second orders.

An appendix contains notes on hyperbolic functions, the intrinsic equations of a curve (in the plane), and the length of a curve in space; also a number of exercises and problems, a brief table of integrals, a useful collection of figures of curves, and a very complete index.

The mechanical features of the book are attractive, and it is conspicuously free from typographical errors.

WALTER B. CARVER.

An Elementary Treatment of the Theory of Spinning Tops and Gyroscopic Motion. By HAROLD CRABTREE. London, Longmans, Green and Company, 1909. 8vo, xii + 140 pp.

"THE object of this book is to bring within the range of the abler Mathematicians at our Public Schools and of First Year undergraduates at the Universities [in England], a subject which has hitherto been considered too difficult for any but the more advanced students in Mathematics, while even they have in many cases failed to derive more pleasure from the study of spinning tops than is contained in submitting the problem to the action of a complicated piece of Mathematical machinery which automatically, though unintelligently, turns out the correct result."

This extract from the preface shows well the character of the greater part of the book. At the present time, the interest of the subject, in view of recent developments, renders such an exposition of value to an engineer. But it may be read with profit by those who wish to obtain some idea of the grade reached by

the English student in his freshman year. It should be stated that the elements of statics and dynamics developed from the Newtonian laws of motion are considered as essential at an early stage as trigonometry, and they are usually commenced before the calculus. They are, however, developed from this point of view with the same degree of rigor which we usually adopt for the earlier parts of pure mathematics. The student thus imbibes the idea that physical phenomena are not isolated facts but rather the consequences of a few general laws.

Although the author starts from the laws of motion, apparently for the sake of logical completeness, it is implicitly assumed that the student is familiar with these laws and that he has had some practice in their applications. From this stage the fundamental property of the top is rapidly obtained:—that the main effect of a couple acting on the axis of the top is a motion of the axis in a plane perpendicular to the plane of the couple, provided the axis is free to move in any direction. An excellent collection of illustrations then follows. Besides the ordinary tops and gyroscopes, familiar to everyone, we find descriptions of Brennan's monorail system, Schlick's method of steadying ships at sea, the steering of torpedoes, the manner in which a cat always manages to land on its feet, the motion of a celt, the game of diabolò, and so on. These descriptions are always followed by explanations sufficient to show the manner in which the gyroscopic effect is involved. There are also numerous other examples taken from familiar experiences with bicycles, engine wheels, hoops, etc. In a few chapters at the end we find the usual mathematical developments.

Some obscurities likely to convey a wrong impression call for mention. On page 36, the text would lead one to infer that the orthogonal projection of an area was the same as the area seen from a point at a finite distance. The explanation on pages 1, 2 of the quasi-rigidity of a rapidly revolving chain or paper disc is unsatisfactory; surely it involves only the ordinary effect of inertia. On page 113, the simple statement that angular momentum is a vector quantity is hardly sufficient for the deduction of the rates of change with respect to moving axes: the author has mentioned no general theorem with respect to vectors.

These are, however, minor points. Mr. Crabtree's effort is decidedly successful and can be recommended as a useful addition to the working library of a department of mathematics,

physics or engineering. The volume is well and fully illustrated with plates and diagrams. An attractive and luminous style has aided the author very considerably in his desire to make the subject clear; in this connection his dedication is not unworthy of notice.

E. W. BROWN.

Dynamics of a Particle and of Rigid Bodies. By S. L. LONEY. Cambridge, University Press, 1909. 8vo. 374 pp. \$4.00.

PROFESSOR LONEY'S mathematical texts are so well known that it is sufficient to say that the high standards of the earlier books have been maintained in the present volume. The text is divided into two equal parts (184 pages each), the first of which is devoted to dynamics of a particle. In the dynamics of a rigid body the author has confined himself chiefly to motion in two dimensions. There is a short chapter on motion in three dimensions, and Lagrange's equations in generalized coordinates are developed for both finite forces and blows. The reader is supposed to have a working knowledge of the calculus, but the differential equations which occur are solved in the text, and an appendix contains a summary of the methods of solution.

It is assumed that the student has previously read a course in elementary dynamics and the present book is mainly a treatment of more difficult problems than occur in a first course. There is a large collection of very interesting examples which furnish excellent material for the cultivation of ability in the solution of problems.

The text has been made as brief as possible and in some places the prerequisite knowledge is greater than any American student is likely to have. For example on page 76 we find that the reader is supposed to be familiar with the (p, r) equations of the conics, where r denotes the distance from the origin O to a point P on the curve, and p is the distance from O to the tangent at P . A recent notice of an American book in an English publication* contains the remark: "In looking through the American text-book a British teacher is almost always puzzled as to what previous equipment the reader is supposed to possess." This statement implies a criticism of recent tendencies in our teaching of mathematics which may or may not be important, depending on the course of study pur-

* *Mathematical Gazette*, March, 1910, p. 215.

sued. If the prospective student of higher mathematics has taken only the ordinary college courses in trigonometry and analytic geometry, he is not provided with machinery to facilitate the work in hand. It is not likely that he can make free use of the hyperbolic functions, and he may have difficulty with oblique axes. The equation of an ellipse referred to a pair of conjugate diameters will mean very little to him if he has never heard of conjugate diameters. To such a student this volume would be very valuable as a source of suggestion for further study in the subjects which ordinarily precede the work in mechanics.

W. R. LONGLEY.

NOTES.

THE annual meeting of the London mathematical society was held on November 10. Reports for the preceding year were presented, and the following officers elected: president, H. F. BAKER; secretaries, A. E. H. LOVE and J. H. GRAOE; treasurer, Sir J. LARMOR; four vice-presidents, and nine other members of the council. The following papers were read at this meeting: By W. D. NIVEN (presidential address), "The relation of mathematics to experimental science"; by G. H. HARDY, "Properties of logarithmico-exponential series"; by G. T. BENNETT, "The double six of lines"; by W. H. YOUNG, "On semi-integrals and oscillating successions of functions," and (with Mrs. YOUNG), "On the existence of a differential coefficient"; by F. TAVANI, "The analytical extension of Riemann's zeta-function"; by T. W. CHADNEY, "The geometrical representation of non real points in space of two and of three dimensions"; by J. E. LITTLEWOOD, "The extension of Tauber's theorem"; by W. H. YOUNG, "A note on the property of being a differential coefficient" by F. B. PIDDUCK, "The stability of rotating shafts"; by J. W. CHAPMAN, "On non-integral orders of summability of series and integrals"; by A. A. ROBB, "Optical geometry of motion"; by A. R. FORSYTH, "Lineo-linear transformations, especially in two variables"; by W. H. YOUNG, "On the condition that a trigonometric series should have the Fourier form."

At the Basel meeting of the Swiss mathematical society Professor Rudio reported on the organization and progress of the committee to superintend the publication of Euler's works. Editors have been selected for all the 43 volumes, and a large part of the material has already been distributed among them. Complete files of the publications of the academies of Berlin and St. Petersburg have been secured, and a complete set of the books of Euler which were published separately was presented to the commission. A set of 165 packages of manuscripts, hitherto in the possession of the St. Petersburg academy, have been received and found to contain a wealth of new material. A large number of portraits have been collected, which will be reproduced in the leading volumes of each series. Teubner, of Leipzig, has been selected as publisher. Three volumes are now in the press and will appear in a few weeks. Further volumes are in preparation and will be issued during 1911. The commission has received subscriptions for 350 sets, and fr. 110,752 in cash.

THE firm of Martin Schilling in Halle announces the following additions to its collection of mathematical models: Series XXX, number 8, plaster model of the surface

$$z = xy \cdot \frac{x^2 - y^2}{x^2 + y^2}, \text{ for which } \frac{\partial^2 z}{\partial x \partial y} \neq \frac{\partial^2 z}{\partial y \partial x} \text{ at the point } (0, 0, 0),$$

by W. FISCHER, with the cooperation of Professors Schilling and Sommer, of the technical school at Danzig; series XXXVI, numbers 1-4, four models to represent affine transformations in the plane and in space, by Professor F. KLEIN; series XXXVII, numbers 1-4, four pasteboard models of regular reentrant polygons, by Professor F. SCHILLING and Dr. O. WIESSING; series XXXVIII, number 1, model to illustrate the theory of the linear line complex, by Professor F. SCHILLING.

THE following university courses in mathematics are announced:

OXFORD UNIVERSITY (Michaelmas term, beginning October 18, 1910). All courses are two hours per week. — By Professor W. ESSON: Analytic theory of plane curves; Synthetic theory of plane curves. — By Professor E. B. ELLIOTT: Sequences and series; Elementary theory of numbers. — By Professor A. E. H. LOVE: Analytic statics and harmonic analysis. — By Professor H. H. TURNER: Mathematical as-

tronomy. — By Dr. J. E. CAMPBELL: Differential equations. — By Mr. H. C. THOMPSON: Integral calculus. — By H. T. GERRANS: Tridimensional rigid dynamics. — By Mr. A. L. DIXON: Hydrostatics. — By Mr. A. L. PEDDER: Problems in pure mathematics. — By Mr. C. E. HASELFOOT: Theory of equations. — By Mr. P. J. KIRKBY: Projective geometry. — By Mr. C. H. SAMPSON: Plane analytic geometry. — By Mr. A. E. JOLLIFFE: Solid geometry. — By Mr. J. W. RUSSELL: Differential calculus. — By Mr. E. H. HAYES: Elementary mechanics.

UNIVERSITY OF PARIS (second semester, beginning March 1, 1911). — By Professor E. PICARD: Functional equations of analysis and of mathematical physics, two hours. — By Professor E. GOURSAT: Ordinary and partial differential equations, two hours. — By Professor P. APPELL: Général laws of motion and analytical mechanics, two hours. — By Professor E. CARTAN: Analysis and mechanics, two hours. — By Professor P. ANDOYER: Program for the higher certificate in astronomy, two hours. — By Professor J. BOUSSINESQ: Dispersion, double and circular diffraction, two hours. — By Professor G. KÖNIGS: General theory of machines, two hours.

COLLEGE OF FRANCE (academic year beginning December 1, 1910). — By Professor J. HADAMARD: Quasi-periodic functions with applications to mechanics, two hours. — By Professor C. JORDAN: Algebraic integers, particularly quadratic numbers, two hours. — By Dr. BRILLOUIN: Elasticity of isotropic and non-isotropic bodies, two hours.

At the University of Glasgow a portrait of Professor WILLIAM JACK was recently unveiled. In commemoration of his fifty years of service as teacher of mathematics, a fund of three hundred pounds has been secured, the income of which will be known as the William Jack prize. It will be awarded to the writer of the most meritorious thesis for the degree of doctor of science in mathematics at the University of Glasgow.

PROFESSORS E. W. HOBSON and Sir J. LARMOR have been appointed moderators for Part II of the Cambridge mathematical tripos in 1911.

THE second award of the Bolyai prize founded in 1905 by the Hungarian academy of sciences (see BULLETIN, volume

12, pages 43, 212) was made to Professor DAVID HILBERT, of the University of Göttingen. The jury of award consisted of Professors König and Rados, of the academy, Professor Mittag-Leffler, of Stockholm, and Professor Poincaré, of Paris.

At the annual meeting of the Royal society of London, the Sylvester medal was awarded to Dr. H. F. BAKER, in recognition of his treatise on abelian functions, and for his edition of the works of Sylvester. The medal is awarded biennially for the encouragement of mathematical research irrespective of nationality.

SIR GEORGE DARWIN has been elected president of the Cambridge philosophical society.

DR. P. KOEBE, of the University of Göttingen, has been appointed associate professor of mathematics at the University of Leipzig.

DR. W. KUTTA has been appointed professor of mathematics at the technical school at Aachen.

DR. J. JOUGUET has been appointed professor of mathematical analysis at the national school of mines at Paris.

PROFESSOR BEPPO LEVI, of the University of Cagliari, has been transferred to the University of Parma.

DR. A. VITERBI, of the University of Pavia, has been promoted to an associate professorship of theoretical geodesy.

PROFESSOR W. S. EICHELBERGER has been appointed director of the U. S. Nautical Almanac Office.

THE examiners in mathematics of the College entrance board for 1911 are Professors R. W. PRENTICE, of Rutgers College, and H. E. HAWKES, of Columbia University, and Mr. E. B. PARSONS, of the Boys High School, Brooklyn, N. Y.

At the University of Nevada Dr. CHARLES HASEMAN has been promoted from an associate professorship to a full professorship of mathematics and mechanics.

MR. I. W. SMITH has been promoted to an assistant professorship of mathematics at the College of Agriculture and Mechanic Arts of the University of North Dakota.

AT Marietta College Professor H. L. COAR has been made full professor of mathematics and Lee lecturer on astronomy.

AT Syracuse University Dr. F. F. DECKER has been promoted to an assistant professorship of mathematics.

MR. R. R. SHUMWAY has been appointed assistant professor of mathematics in the University of Minnesota.

AT the State University of Iowa Dr. R. P. BAKER has been promoted to an assistant professorship of mathematics.

DR. A. M. HILTEBEITEL has been appointed instructor in mathematics at the University of Pennsylvania.

MR. W. S. PEMBERTON, of the University of Missouri, has been appointed professor of mathematics at Wheaton College, Wheaton, Ill.

PROFESSOR MAURICE LÉVY, of the College of France, died in October, at the age of 72 years.

COOPER D. SCHMITT, professor of mathematics in the University of Tennessee for twenty-one years, and dean of the University, died December 9, at the age of 51 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

- AMSON (E.). Eine zeichnerisch wohl verwertbare Konstruktionsweise des allgemeinen Kegelschnittes. (Progr.) München, 1910. 8vo. 19 pp.
- BARUCH (A.). Ueber die Differentialrelationen zwischen den Thetafunktionen eines Arguments. (Diss.) Halle a. S., 1910.
- CARVALLO (E.). Méthode pratique pour la résolution numérique complète des équations algébriques et transcendentes. 3e édition. Paris, 1910. 4to.
- CHRISTOFFEL (E. B.). Gesammelte mathematische Abhandlungen. Unter Mitwirkung von A. Krazer and G. Faber herausgegeben von L. Maurer. 1ter Band, mit einer Biographie E. B. Christoffels. Leipzig, Teubner, 1910. 8vo. 16+382 pp. M. 18.00
- . Gesammelte mathematische Abhandlungen. 2ter Band. Leipzig, Teubner, 1910. 8vo. 3+343 pp. M. 16.00
- CICCO (S. DE). Sulla protogenesi dei processi matematici. Napoli, 1910. L. 2.00
- DARBOUX (G.). Leçons sur les systèmes orthogonaux et les coordonnées curvilignes. 2e édition, complétée, Paris, Gauthier-Villars, 1910. 8vo. 576 pp. Fr. 18.00
- ENCYCLOPÉDIE des sciences mathématiques pures et appliquées. Tome I, volume 2, fascicule 2: Propriétés générales des corps et des variétés algébriques, par G. Landsberg, J. Hadamard, et J. Kurschák. Leipzig, Teubner, 1910. M. 3.00
- FESTSCHRIFT zur Feier des 100 Geburtstages Eduard Kummers. Mit Briefen an seine Mutter und an L. Kronecker. Herausgegeben vom Vorstande der Berliner Mathematischen Gesellschaft. Mit einem Bildnis E. Kummers. (Abhandlungen zur Geschichte der mathematischen Wissenschaften, Heft XXIX.) Leipzig, Teubner, 1910. 8vo. M. 4.00
- FICK (E.). Anfangsgründe der analytischen Geometrie des 4fach ausgedehnten Raumes. 2ter Teil. (Progr.) Neuberg a. D., 1910. 8vo. 31 pp.
- HARDY (G. H.). Orders of infinity. The "Infinitarcalcul" of Paul Du Bois-Reymond. (Cambridge tracts in mathematics and mathematical physics. No. 12.) London, Cambridge University Press, 1910. 8vo. 62 pp. 2s. 6d.
- HELM (G.). Die Grundlehren der höheren Mathematik. Zum Gebrauch bei Anwendungen und Wiederholungen zusammengestellt. Leipzig, 1910. 8vo. 15+419 pp. Cloth. M. 14.20
- IN MEMORIAM Irving Stringham. Professor Stringham's services to the university. William T. Reid. Irving Stringham. Mellen W. Haskell. Memorial resolutions: Regents of the University of California. Academic council. Students of the university. Faculty Club. Harvard Club of San Francisco. Berkeley Club. Berkeley, University of California, 1910.
- FRISTNER (C.). Die einem homogenen linearen Differentialgleichungssystem zugeordneten Systeme. (Diss.) Freiburg i. Br., 1910.
- REID (L. W.). The elements of the theory of algebraic numbers. With an introduction by David Hilbert. New York, Macmillan, 1910. 8vo. 15+461 pp. Cloth. \$3.50
- ROTHE (R.). Ernst Gerland zum Gedächtniss. Clausthal, 1910.

VEHLEN (O.) and YOUNG (J. W.). Projective geometry. Volume I. Boston, Ginn, 1910. 8vo. 10+352 pp. Cloth. \$4.00

YOUNG (J. W.). See VEHLEN (O.).

II. ELEMENTARY MATHEMATICS.

ANDREINI (A. L.). Tavole dei logaritmi con tre e con quattro decimali. Seconda edizione riveduta. Livorno, Giusti, 1910. 16mo. 8+58 pp.

BAITEN (R.) and MATWALD (W.). Kurzgefasstes Lehrbuch der Mathematik für Seminare und Präparandenanstalten. Nach den Lehrplänen von 1901 bearbeitet. 3te verbesserte Auflage. Leipzig, Teubner, 1910. 8vo. 10+240 pp. Cloth. M. 2.60

BARNARD (S.) and CHILD (J. M.). Key to a new algebra. Vol. I. London. 1910. 6s. 6d.

BELL (R. J. T.). An elementary treatise on coordinate geometry of three dimensions. London, Macmillan, 1910. 8vo. 372 pp. 10s.

BENDT (F.). Grundlege der Differential- und Integralrechnung. 4te verbesserte Auflage. Leipzig, Weber, 1910. 8vo. 16+268 pp. Cloth. M. 3.00

BETTAZZI (R.). Lezioni di calcolo integrale. Torino, 1910. L. 5.00

BRENKE (W. C.). Textbook on advanced algebra and trigonometry, with tables. New York, Century, 1910. 8vo. 7+345 pp. \$2.00

BREWSTER (C. W.). See SANDERSON (F. W.).

CHILD (J. M.). See BARNARD (S.).

CHRAMER (H.). Der mathematische Unterricht an den höheren Schulen im Grossherzogtum Baden. (Abhandlungen, herausgegeben von F. Klein, Band II, Heft 4.) Leipzig, Teubner, 1910. 8vo. 4+48 pp. M. 1.60

DESCHAMPS (J.). Notes sur la géométrie analytique. Applications de nos identités fondamentales. Paris, 1910.

DINTAL (E.). Der mathematische Unterricht an den Gymnasien. (Berichte über den mathematischen Unterricht in Österreich, Heft 3.) Wien, Holder, 1910.

DOLINSKI (M.). Algebra und politische Arithmetik. 2te, vermehrte und verbesserte Auflage. Wien, 1910. M. 5.00

DÜSING (K.). Die Elemente der Differential- und Integralrechnung, in geometrischer Methode dargestellt. Ausgabe B: mit zahlreichen Beispielen aus der technischen Mechanik von E. Preger. 2te Auflage. Hannover, Janecke, 1910. 8vo. 11+101 pp. M. 1.90

ELSON (W. H.). See SHORT (R. L.).

ERDMANN (K.). Anfangsgrunde der ebenen Geometrie. 2te, umgearbeitete Auflage. Dresden; Bleyl und Kaemmerer, 1911. 8vo. 8+283 pp. Cloth. M. 4.40

FARMER (R. C. and M. M.). Four-figure logarithms on a new graphic system, dispensing with interpolations. New York, Longmans, 1910. 12mo. Limp cloth. \$0.20

FERRARIS (P.). Elementi di geometria, ad uso delle scuole medie. Vol. I. Torino, Paravia, 1911. 16mo. 192 pp. L. 3.00

FRANCHIS (M. DE). Elementi d'algebra, ad uso dei licei. Vol. II: per le 2e e 3e classi licei. Palermo, Sandron, 1910. 16mo. 154 pp. L. 2.00

GECK (E.). Der mathematische Unterricht an den höheren Schulen im Königreich Württemberg. (Abhandlungen, herausgegeben von F. Klein, Band II, Heft 3.) Leipzig, Teubner, 1910. 8vo. 4+104 pp. M. 1.60

GRANVILLE (W. A.). See SMITH (P. F.).

- GRÜNBAUM (H.). Der mathematische Unterricht an den deutschen mittleren Fachschulen der Maschinenindustrie. (Abhandlungen, herausgegeben von F. Klein, Band IV, Heft 1. Mit einem Einführungswort von P. Stäckel.) Leipzig, Teubner, 1910. 8vo. 16+100 pp. M. 2.60
- KOMMERELL (V.). Raumgeometrie: Stereometrie und darstellende Geometrie. Tübingen, Laupp, 1910. 8vo. 8+196 pp. M. 2.60
- LAZZERI (G.). Manuale di trigonometria sferica. Seconda edizione riveduta. Livorno, Giusti, 1911. 16mo. 4+95 pp. L. 1.00
- LIEFMANN (W.). Die organisation des mathematischen Unterrichts an den höheren Knabenschulen in Preussen. (Abhandlungen, herausgegeben von F. Klein, Band I, Heft 2.) Leipzig, Teubner, 1910. 8vo. 8+204 pp. M. 5.00
- LINNICH (M.). Lehr- und Übungsbuch für den Unterricht in der Arithmetik und Algebra mit einem Anhang für den Unterricht in der analytischen Geometrie. Leipzig, Freytag, 1910. M. 2.50
- MAIWALD (W.). See BALLEN (R.).
- MANGUSE (W. P.). See SCHULTZE (A.).
- MEIGEN (F.). Lehrbuch der Trigonometrie. (Technische Lehrhefte.) 2te, vermehrte und verbesserte Auflage. Leipzig, Gebhardt, 1910. 8vo. 3+77 pp. M. 2.40
- MERCER (J. W.). The calculus for beginners. New York, Putnam, 1910. 12mo. 14+440 pp. Cloth. \$2.00
- MÜLLER (H. H.). See REINHARDT (W.).
- NIENGLÓWSKI (B.). Deuxième année de géométrie. Premier cycle. Paris, Delagrave, 1910. 18mo. 154 pp. Fr. 2.25
- REINHARDT (W.) und MÜLLER (H. H.). Lehrbuch für den mathematischen Unterricht auf Grund der neuen Lehrpläne. 2ter Teil (Nachtrag): Trigonometrie. Frankfurt a. M., Auffarth, 1910. 8vo. 4+56 pp. M. 1.00
- SALAZAR E IBÁÑEZ (M.). Problemas matemáticos. Madrid, Marso, 1910. 8vo. P. 7.00
- SANDERSON (F. W.) and BREWSTER (C. W.). A geometry for schools. London, Cambridge University Press, 1910. 8vo. 346 pp. 3s.
- SCHAEFFER (A.). 1400 mathematische Abiturienten-Aufgaben mit 700 Lösungen und Anleitungen zur Lösung. Zabern, Fuchs, 1910. 8vo. 16+240 pp. Cloth. M. 3.80
- SCHANE (I. A.). Lehrbuch der Mathematik zum Gebrauch in höheren Mädchenschulen. 2ter Teil, 2tes Heft: Geometrie. Berlin, Oemigke, 1910. 8vo. 192 pp. Cloth. M. 2.80
- SCHMIDT (J.). Lehrbuch der Elementarmathematik. Ausgabe für Gymnasien und Realgymnasien. 1ter Band. Wien, Hölder, 1910. 8vo. 7+336 pp. Cloth. M. 2.80
- . Lehrbuch der Elementarmathematik. Ausgabe für Realschulen. 1ter Band. Wien, Hölder, 1910. 8vo. 7+387 pp. Cloth. M. 3.30
- SCHNELL (H.). Der mathematische Unterricht an den höheren Schulen im Grossherzogtum Hessen. (Abhandlungen, herausgegeben von F. Klein, Band II, Heft 5.) Leipzig, Teubner, 1910. 8vo. 6+51 pp. M. 1.60
- SCHULTZE (A.) and MANGUSE (W. P.). Answers to elements of algebra. New York, Macmillan, 1910. 12mo. 38 pp. Paper. \$0.20
- SCHUSTER (A.). Einführung in die elementare Mathematik. (Sammlung Kösel, Band 33.) München, Kösel, 1910. M. 1.00

- SHORT (R. L.) and ELSON (W. H.). Secondary school mathematics, Book 1. Boston, Heath, 1910. 12mo. 182 pp. \$1.00
- SMITH (C.). An elementary treatise on conic sections by the methods of co-ordinate geometry. New edition, revised and enlarged. London, Macmillan, 1910. 8vo. 460 pp. 7s. 6d.
- SMITH (P. F.) and GRANVILLE (W. A.). Elementary analysis. Boston, Ginn, 1910. 12mo. 223 pp. \$1.50
- TESTI (G. M.). Elementi di geometria. Tredicesima edizione, modificata e corretta. Livorno, Giusti, 1911. 16mo. 9+239 pp. L. 1 75
- TIMMERDING (H. E.). Die Mathematik in den physikalischen Lehrbüchern. (Abhandlungen, herausgegeben von F. Klein, Band III, Heft 2.) Leipzig, Teubner, 1910. 8vo. 6+112 pp. M. 2.80
- WEILL (A.). Sammlung graphischer Aufgaben für den Gebrauch an höheren Schulen. I. Mathematik. Gebweiler, Boltz, 1910.
- WELLS (W.). Second course in algebra. Boston, Heath, 1910. 12mo. 6+283 pp. Cloth. \$1.00
- WIEBERTNER (H.). Der mathematische Unterricht an den höheren Lehranstalten im Königreich Bayern. Mit einem Einführungswort von P. Treutlein. (Abhandlungen, herausgegeben von F. Klein, Band II, Heft 1.) Leipzig, Teubner, 1910. 8vo. 14+85 pp. M. 2.40
- WILCKEN (F.). Mathematische Tabellen. Leipzig, Degener, 1910. 8vo. 25 pp. M. 0.25
- WITTING (A.). Der mathematische Unterricht an den Gymnasien und Realanstalten im Königreich Sachsen. (Abhandlungen, herausgegeben von F. Klein, Band II, Heft 2.) Leipzig, Teubner, 1910. 8vo. 12+78 pp. M. 2.20
- WOTRUBA (R.). Analytische Geometrie der Ebene. Ein Leitfaden für Studierende technischer Mittelschulen. Leipzig, Gebhardt, 1910. 8vo. 6+50 pp. M. 2.08
- ZIMMERMANN (H.). Rechen tafel nebst Sammlung häufig gebrauchter Zahlenwerte. 6te Auflage. Ausgabe A, ohne besondere Quadrattafeln. Berlin, Ernst, 1910. 8vo. 34+204 pp. Cloth. M. 5.00
- . Rechen tafel nebst Sammlung häufig gebrauchter Zahlenwerte. Ausgabe B, mit Anhang, enthaltend Quadrattafeln. Berlin, Ernst, 1910. 8vo. 6+204+20 pp. Cloth. M. 6.00

III. APPLIED MATHEMATICS.

- AUERBACH (M.). Elementary course in graphic mathematics. Boston, Allyn, 1910. 12mo. 3+54 pp. \$0.35
- BOREL (E.). See PAINLEVÉ (P.).
- BOUGUERET (A. et P.). Cours élémentaire de géométrie descriptive. 1re partie. Paris, Paulin, 1910. 8vo. 14 pp.
- COTSWORTH (M. B.). Direct calculator. Tables pour les calculs de tous genres. Paris, 1910. Fr. 25.00
- CRANE (C.). Lehrbuch der Ballistik. 1ter Band. Äussere Ballistik. 2te, vollständig umgearbeitete Auflage des "Compendiums der theoretischen äusseren Ballistik" von 1896. Leipzig, Teubner, 1910. 8vo. 4+464 pp. Cloth. M. 20.00
- . Lehrbuch der Ballistik. 4ter Band: Atlas für Tabellen, u.s.w. Leipzig, Teubner, 1910. 5+81 pp. Cloth. M. 14.00
- DAVAGLIO (G.). Nuovi principi di dinamica lineare. Saggio. Parte II. Bergamo, Istituto italiano d'arti grafiche, 1910. 8vo. pp. 66-153. L. 3.00

- DELP (J. D.). Comprehensive modernized rapid calculator. Springfield, Mo., Delp, 1910. 12mo. 100 pp. Cloth. \$0.75
- DER TUNZELMANN (G. W.). A treatise on electrical theory and the problem of the universe. Considered from the physical point of view, with mathematical appendices. London, 1910. 8vo. 685 pp. Cloth. \$4.50
- DUNCAN (R. H.). Practical curve tracing, with chapters on differentiation and integration. London, 1910. 5s.
- DUTHEIL (P.). See TEXIER (A.).
- ENNIS (W. D.). Applied thermodynamics for engineers. New York, Van Nostrand, 1910. 8vo. 8+438 pp. \$4.50
- ENCYCLOPÄDIE der mathematischen Wissenschaften. Band VI 2, Heft 3: GINSEL und WILKENS, Theorie der Finsternisse; GINSEL, Chronologie; HERGLOTZ, Bahnbestimmung der Planeten und Kometen; von NIESSL, die Bestimmung der Meteorbahnen im Sonnensystem. Leipzig, Teubner, 1910. M. 3.60
- der mathematischen Wissenschaften. Band VI 1B, Heft 2: Die Schwerkraft und die Massenverteilung der Erde. Von F. R. HOLMERT. Leipzig, Teubner, 1910.
- FICHOT (R.). See POINCARÉ (H.).
- FREYBERGER (H.). Zentral-Perspektive. (Sammlung Götschen, 57.) Neu bearbeitet von J. VONDERLINN. Neudruck. Leipzig, Göschen, 1910. 8vo. 137 pp. Cloth. M. 0.80
- HAUSNER (R.). Darstellende Geometrie. 1ter Teil: ebenflächige Gebilde. (Sammlung Götschen, 142.) 2te, vermehrte und verbesserte Auflage. 3te, neu durchgesehene Abdruck. Leipzig, Göschen, 1910. 8vo. 207 pp. Cloth. M. 0.80
- HECK (R. C. H.). Notes on the graphics of machine forces. New York, Van Nostrand, 1910. 8vo. 42 pp. Boards. \$1.00
- HOPKINSON (B.). Vibrations of systems having one degree of freedom. (Cambridge engineering tracts.) New York, Putnam, 1910. 8vo. Paper. \$0.75
- JANET (P.). Premiers principes d'électricité industrielle. Ouvrage couronné par l'académie des sciences. 6e édition, revue et corrigée. Paris, Gauthier-Villars, 1910. 8vo. 8+282 pp. Fr. 6.00
- KALÁHNE (A.). Grundzüge der mathematisch-physikalischen Akustik. 1ter Teil. (Mathematisch-physikalische Schriften für Ingenieure und Studierende, 11.) Leipzig, Teubner, 1910. 8vo. 7+130 pp. M. 3.60
- KORN (A.). Freie und erzwungene Schwingungen. Eine Einführung in die Theorie der linearen Integralgleichungen. Leipzig, Teubner, 1910. M. 5.60
- LINDOW (M.). Die Anwendung der Differentialrechnung auf das technische Zeichnen. Jena, 1910. M. 2.00
- MACH (E.). Populär-wissenschaftliche Vorlesungen. 4te Auflage. Leipzig, 1910. M. 6.80
- MATTER (E.). Die Symmetrie der gerichteten Größen, besonders der Kristalle. (Progr.) LINS, 1910.
- PAINLEVÉ (P.) et BOREL (E.). L'aviation. Paris, Alcan, 1910. 16mo. 8+270 pp. Fr. 3.50
- PEDDLE (J. B.). The construction of graphical charts. New York, McGraw-Hill, 1910. 8vo. 8+109 pp. Cloth. \$1.50
- POINCARÉ (H.). Leçons de mécanique céleste. Tome III. Théorie des marées, par R. FICHOT. Paris, 1910. Fr. 16.00

- PRAYON (E.). Etude sur les hélices propulsives. En particulier, les hélices aériennes. Théorie complète, calcul, tracé, constructions, à l'aide de formules élémentaires nouvelles et exactes. Paris, Dunod, 1910. 8vo. 14+92 pp. Fr. 3.75
- RAYMOND (W. G.). Railroad field geometry. New York, Wiley, 1910. 16mo. 9+246 pp. Morocco. \$2.00
- SCHWAHN (P.). Mathematische Theorie der astronomischen Finsternisse. (Mathematisch-physikalische Schriften für Ingenieure und Studierende, herausgegeben von E. Jahnke. 8tes Bandchen.) Leipzig, Teubner, 1910. 8vo. 6+128 pp. Cloth. M. 3.80
- SIEGER (B.). Die Beugung einer ebenen elektrischen Welle an einem Schirm von elliptischem Querschnitt. (Progr.) Würzburg, 1910. 8vo. 39 pp.
- SORBAU (R.). Variations du coefficient de SV^2 dans le déplacement orthogonal des plans. (Introduction à la méthode des petits modèles de voilures.) Paris, Dunod, 1909. 8vo. 18 pp.
- STARK (J.). Prinzipien der Atomdynamik. 1ter Teil: Die elektrischen Quanten. Leipzig, Hirzel, 1910. 8vo. 10+124 pp. M. 4.00
- STONEY (G. A.). The theory and practice of perspective. Oxford, 1910. 10s.
- TEXIER (A.) et DUTHIEL (P.). Eléments de mécanique générale et appliquée. 3e édition, complètement remaniée, remise à jour et conforme au programme de 1909. Paris, Nathan, 1910. 16mo. 8+180 pp.
- TOMES (B. A.). A first course in practical mathematics. London, Blackie, 1910. 8vo. 188 pp. 1s. 6d.
- VOIGT (W.). Lehrbuch der Kristallphysik, mit Ausschluss der Kristalloptik. (Teubner's Sammlung, 34.) Leipzig, Teubner, 1910. 8vo. 24+964 pp. Cloth. M. 32.00
- WALLERANT (F.). La cristallographie géométrique. Paris, 1910. 8vo. Fr. 1.50
- WEBER (R.). Beispiele und Übungen aus Elektrizität und Magnetismus. Leipzig, Teubner, 1910. 8vo. 8+330 pp. Cloth. M. 5.25
- WHITTAKER (E. T.). A history of the theories of ether and electricity from the age of Des Cartes to the close of the nineteenth century. London, Longmans, 1910. 8vo. 16+475 pp. Cloth. 12s. 6d.



THE FOURTH REGULAR MEETING OF THE SOUTHWESTERN SECTION.

THE fourth regular meeting of the Southwestern Section of the Society was held at the University of Nebraska, Lincoln, Nebraska, on Saturday, November 26, 1910. The following members of the Society were present:

Professor W. C. Brenke, Professor E. W. Davis, Professor E. P. R. Duval, Professor C. C. Engberg, Professor E. R. Hedrick, Professor O. D. Kellogg, Miss H. H. MacGregor, Professor W. H. Roever, Professor O. Schmiedel, Professor J. W. Young.

The morning session was opened at 10 A. M. and the afternoon session at 2 P. M., Professor Davis presiding. St. Louis was fixed as the next place of meeting of the section, and the following program committee elected: Professor Roever (chairman), Professor Brenke, Professor Kellogg (secretary). On Friday evening a smoker was held at the home of Professor Davis, at which some of the problems of the mathematical departments of the state universities were informally discussed, and the attending members lunched together on Saturday.

The following papers were presented at this meeting:

(1) Professor G. A. MILLER: "Abstract definition of all the substitution groups whose degrees do not exceed seven."

(2) Dr. ELIZABETH R. BENNETT: "Primitive groups of degree 20."

(3) Professor W. C. BRENKE: "On the series of zonal harmonics."

(4) Professor A. B. FRIZELL: "On certain transfinite permutations."

(5) Professor A. D. PITCHER: "Properties of certain classes of sequences."

(6) Professor E. W. DAVIS: "Imaginarities on a cubic" (preliminary communication).

(7) Dr. LOUIS INGOLD: "Curves in a function space."

(8) Dr. LOUIS INGOLD: "Surfaces in a function space."

(9) Professor FLORIAN CAJORI: "Horner's method of approximation anticipated by Ruffini."

(10) Professor W. H. ROEVER: "Southerly deviation of falling bodies. Second paper."

(11) Professor J. W. YOUNG: "On fundamental regions for cyclical groups of linear transformations on two complex variables."

(12) Professor O. D. KELLOGG: "Green's integral for multiply connected regions."

(13) Professor E. R. HEDRICK: "On assemblages with closed derivatives."

Dr. Bennett was introduced by Professor Davis. The papers of Professors Miller, Frizell, and Pitcher were read by title. Dr. Ingold's papers were presented by Professors Kellogg and Hedrick, respectively, and Professor Cajori's paper by Professor Davis. Abstracts of the papers follow below.

1. In addition to giving a list of possible abstract definitions of the substitution groups whose degrees do not exceed seven, and an explanation of these and other possible abstract definitions, Professor Miller proves several general theorems relating to such definitions. Among these are the following: If n operators of order two are such that the product of any three of them is also of order two, they generate a group which involves an abelian subgroup composed of half the operators of the group, and each of the remaining operators is of order two and transforms each operator of this abelian subgroup into its inverse. Moreover, if this abelian group involves no more than $n - 1$ invariants, the entire group of twice its order can always be generated by n such operators of order two. The necessary and sufficient conditions that the augmented right co-sets

$$H + HS_2 + HS_3 + \dots + HS_p$$

constitute a group are that they include the product of any two of the operators $S_2, S_3, \dots, S_p$ and that they also include the inverse of each operator in these co-sets. New abstract definitions for the general symmetric and alternating group of degree n were also given.

2. Dr. Bennett proves in this paper certain theorems concerning the transitive constituents of the maximal subgroup G_1 of a primitive group $\mathcal{G}$. The primitive groups of degree 20 are then determined and it is shown that only four such groups exist. These are the symmetric and alternating groups of degree 20 and the two well-known Mathieu groups of degree $p + 1$ and orders $p(p^2 - 1)$ and $\frac{1}{2}p(p^2 - 1)$.

3. In Professor Brenke's paper the following theorem is obtained concerning the series

$$a_1 X_1 + a_2 X_2 + \dots + a_n X_n + \dots,$$

where X_n denotes the Legendrian polynomial of degree n in x ;

The above series converges uniformly in any interval (c_1, c_2) lying between -1 and $+1$ and not including either of those points, provided that

(1) it converges at one point of (c_1, c_2) ;

(2) the series $\sum_1^{\infty} A_n X_n$, where

$$A_n = \frac{n(n-1)}{2n-1} a_{n-1} - \frac{(n+1)(n+2)}{2n+3} a_{n+1},$$

converges uniformly in (c_1, c_2) ;

(3) $\lim_{n \rightarrow \infty} a_n / \sqrt{n} = 0$.

If the sum of the series be denoted by $f(x)$, this function will have in (c_1, c_2) a continuous first derivative given by the equation

$$f'(x) = \frac{2a_1}{3(1-x^2)} + \frac{1}{x^2-1} \sum_1^{\infty} A_n X_n.$$

Example: Applied to the series

$$\frac{3}{2} X_1 + \frac{7}{3.4} X_2 + \frac{11}{5.6} X_3 + \dots,$$

this gives

$$f'(x) = \frac{1}{1-x^2}, \quad \text{hence} \quad f(x) = \frac{1}{2} \log \frac{1+x}{1-x}.$$

4. In this paper Professor Frizell develops a procedure for producing sets of permutations of the elements of an ω series which exhibit all ordinal types of the second class.

The prime numbers form an ω series; the products of two, three, $\dots$, n primes form series of types $\omega^2, \omega^3, \dots, \omega^n$ respectively. Thus the set of all products of primes (i. e., the series

of natural numbers) may be arranged in type ω^* . Repetitions of this process combined with that of finite permutation explained in a previous paper yield permutations of the set of all natural numbers arranged in a series of type Ω .

The paper will appear in the *Science Bulletin* of the University of Kansas.

5. The General Analysis of Moore relates to certain properties of classes of real valued functions of a general variable. In this paper Professor Pitcher discusses certain classes of sequences (functions the range of whose variable is the denumerable class $1, 2, \dots, n, \dots$), showing which of the above mentioned properties are possessed by each class. The classes discussed are (a) all convergent series, (b) all convergent sequences, (c) all ultimately constant sequences. Each of these classes possesses properties which make the methods and theorems of general analysis applicable to it. The so-called * extension of each of the above classes is computed.

6. Professor Davis considers the intersections with the cubic of a pencil of lines passing through a point upon the cubic. The imaginary intersections are represented by vectors along lines of the pencil joining points on a certain auxiliary cubic to points on a sextic complementary (in Poncelet's sense) to the original cubic. Through every point in the plane pass three such auxiliary cubics.

7. In Dr. Ingold's first paper use is made of the well known analogy between ordinary vectors and functions of a real variable x on an interval, say $a \leq x \leq b$; functions $f(x; s)$ are considered which involve a parameter s besides the variable x ; the function f is regarded as the analogue of the defining vector of a curve in ordinary space. The extension of Frenet's formulas is used to define a set of functions analogous to the normal vectors of a curve. The usual vector formulas for curves in an n -dimensional space are then reproduced in terms of the function f and its derivatives with respect to the parameter s .

8. The second paper of Dr. Ingold contains a discussion of functions involving several parameters $u_1, u_2, \dots, u_n$ besides the variable x on the interval $a \leq x \leq b$. A function

$f(x; u_1, u_2, \dots, u_n)$ is regarded as the analogue of the defining vector of an n -dimensional surface. Functions analogous to normals to a surface are defined in terms of the first and second partial derivatives of f with respect to the parameters $u_1, \dots, u_n$. The investigation is then limited to the case in which, for given values of the parameters u , the normals are all numerical multiples of the same function N and in which the derivatives of N with respect to the parameters are all linearly expressible in terms of the derivatives of f . With these restrictions on f the principal formulas of the differential geometry of n -dimensional surfaces in a $(n+1)$ -dimensional (euclidean) space, are obtained in terms of f and its derivatives.

9. Professor Cajori shows that Paolo Ruffini published in 1804 at Modena in Italy a monograph which describes a method of approximation to the real roots of numerical equations which is almost identical with the method given by W. G. Horner in 1819 in England. The difference is unimportant: the coefficients of the transformed equation appear, with Ruffini in a column, with Horner along a diagonal line, slanting upward from left to right.

10. In computing the southerly deviation of a falling body, one can not, with impunity, neglect the curvature of the lines of force in the neighborhood of the point P_0 (fixed with respect to the rotating earth) from which the body falls. Unless the lines of force of the rotating field are rectilinear (which, in particular, is the case when the field is constant), a plumb-line P_0R supported at P_0 does not coincide with the tangent t at P_0 to the line of force of the rotating field which passes through P_0 . Some writers, Gauss for instance, measure the southerly deviation from the cone of revolution generated by t . In experiments, on the other hand, the southerly deviation is measured on the equipotential surface (of the rotating field) which passes through the plumb-bob R , from the circle of latitude generated by R . Therefore, in order to compare the results of experiment with those of theory, the latter method must be used. This method is the same as that described by Professor Roever in an earlier paper.*

In his present paper, a formula for the southerly devi-

* BULLETIN, vol. 16 (1910) p. 228.

ation (hereafter denoted by S. D.) is expressed in terms of the potential function $U_1 = f_1(r, z)$ (where r and z denote the distances of a general point from the axis of rotation OZ and the plane of the equator of the earth) of the earth's fixed field of force. The formula is of the form

$$(1) \quad \text{S.D.} = Ah^2 + \dots,$$

where $h = P_0R$ is the distance through which the body falls and A is a function of the earth's angular velocity of rotation ω and the first and second derivatives of f_1 for the point P_0 (r_0, z_0).

If, in particular, we assume the (rotating) field of force in the neighborhood of P_0 to be constant and of intensity g ,

$$f_1 = f_2 - \frac{1}{2}\omega^2 r^2, \quad \text{where} \quad f_2 = -g \cos \phi \cdot r - g \sin \phi \cdot z,$$

in which the constant ϕ is the astronomical latitude at P_0 . For this assumption formula (1) becomes

$$(2) \quad \text{S.D.} = \frac{2}{3} \sin \phi \cos \phi \frac{\omega^2}{g} h^2,$$

or

$$\frac{1}{3} \sin \phi \cos \phi g \omega^2 t^4, \quad \text{since} \quad h = \frac{1}{2} g t^2 + \dots$$

This is the formula given by Gauss* and later writers.

If, on the other hand,

$$f_1 = \frac{M}{\rho} \left[1 + \frac{\epsilon}{3} \frac{\rho_1^2}{\rho^2} \frac{r^2 - 2z^2}{\rho^2} \right], \dagger$$

where M is the mass of the earth, $\rho = \sqrt{r^2 + z^2}$, ρ_1 the mean radius of the earth and ϵ the constant .00167, formula (1) takes the form

$$(3) \quad \text{S.D.} = .0055 \sin 2\phi_0 \cdot \frac{h^2}{\rho_0},$$

where ϕ_0 is the geocentric latitude of P_0 and ρ_0 is ρ for P_0 .

* Gauss, Werke, vol. 5, 1867, p. 502.

† A formula given by H. Poincaré, Figures d'équilibre d'une masse fluide (1902), p. 107.

This formula gives values which are four and one half times as great as those of formula (2).

11. The purpose of Professor Young's note is to call attention to a very simple method of obtaining fundamental regions for cyclical groups of linear fractional transformations on two complex variables. The simplicity of the method is due to the fact that the determination of a fundamental region for a group of the specified kind is made to depend on the construction of a fundamental region for a simply isomorphic group on a single complex variable. The method, moreover, may be readily extended to the case in which the number of variables is n and to certain restricted types of groups which are not cyclical and not linear.

12. Continuing the work which appeared in the *Transactions* of January, 1908, Professor Kellogg makes an immediate extension to multiply connected regions. By means of the results Green's integral formula

$$u(x, y) = \frac{1}{2\pi} \int u(s) \frac{\partial G}{\partial n} ds$$

is rigorously established without any presuppositions concerning $u(x, y)$. From this criteria are obtained for the unique determination of harmonic functions by their boundary values in the case that these are not continuous.

13. At the last meeting of the Section, Professor Hedrick proved that a few important theorems of the theory of point sets hold true for any compact set of objects for which the first derived set is closed. In the present paper the hypothesis is added that any element of the set is enclosable in the interior of a family of diminishing assemblages in a uniform manner. With this hypothesis practically all of the fundamental theorems of point sets are shown to hold for general sets of objects; the list of theorems includes the general Heine-Borel theorem, the Cantor-Bendixon theorem, the Cauchy fundamental theorem, the oscillation theorem, and many others. No distance notion is used; those theorems which usually involve distance are reworded to avoid that concept.

O. D. KELLOGG,
Secretary of the Section.

UNIVERSITY COURSES IN MATHEMATICS AND THE MASTER'S DEGREE.

PROVISIONAL REPORT OF THE AMERICAN SUBCOMMITTEE OF THE INTERNATIONAL COMMISSION ON THE TEACHING OF MATHEMATICS.*

I. *Preliminary Remarks.*

IN preparing this report the committee has found the greatest difficulty in securing reliable information on the various topics considered. Nothing of the kind has been attempted since Cajori's report of 1890, "On the Teaching and History of Mathematics in the United States."† Valuable as this is, it covers only the period of the beginning of graduate study; and except for two brief statistical notes in the Reports of the U. S. Commissioners of Education and scattered items of information in the BULLETIN of the American Mathematical Society, there exists no published information on the development of graduate instruction in mathematics since 1890. The Committee has, therefore, been compelled to gather its material almost entirely at first hand, relying mainly on personal correspondence and on statements contained in catalogues and other publications of American universities. It is needless to say that only a limited amount of information can be obtained in the former way, while as for the latter, it is too frequently the case that the facts desired are either very hard to get at, or are not given at all.

Over thirty colleges and universities in the United States offer graduate work in mathematics, but less than fifteen have given a doctor's degree in that subject within the last five years. The following twenty-four offer courses of an advanced character and report three or more graduate students for the year

* *International Commission on the Teaching of Mathematics.* Committee XIV.—Graduate Work in Universities. Subcommittee 1—Courses of Instruction and the Master's Degree. Chairman, D. R. CURTISS, Northwestern University; EDWARD KASNER, Columbia University; A. C. LUNN, University of Chicago. This report was submitted to the American Commissioners in November, 1910. The report of Subcommittee 3 was published in the November, 1910, BULLETIN.

† U. S. Bureau of Education, Circular of Information No. 3, 1890.

1908-09: Bryn Mawr, California, Chicago, Cincinnati, Clark, Colorado, Columbia, Cornell, Harvard, Illinois, Indiana, Iowa, Johns Hopkins, Michigan, Missouri, Nebraska, Northwestern, Pennsylvania, Princeton, Leland Stanford, Syracuse, Virginia, Wisconsin, Yale. To this somewhat arbitrary list the greater part of the statistics of this report refer, but even in the field thus limited it has been impossible to obtain entirely reliable information on all the points covered. Accordingly, though the numerical statements here given are believed to be in the main correct, it has seemed best not to give detailed statistics for any particular institution.

II. *Historical Sketch.*

Genuine graduate work in mathematics may be said to date from the founding of Johns Hopkins University, and the assumption of his duties there by Sylvester, in the year 1876-1877. It is true that courses of an advanced character had previously been given in various institutions, notably at Harvard, where Benjamin Peirce taught for nearly fifty years and exerted a profound influence on American mathematical development; but this earlier work was never of an exclusively graduate character, and was carried on with frequent interruptions due to the lack of properly prepared students. Johns Hopkins was founded primarily as a graduate school and that department has always overshadowed the college. Its great success proved a powerful stimulus to advanced work along all lines in this country and led ultimately to the founding of separate graduate departments in most of the leading American universities.

The group of eight or ten whom Sylvester soon gathered about him, and their successors, carried something of their master's enthusiasm with them when they went out to other institutions, and in the course of a few years advanced instruction in mathematics was begun in a number of other colleges and universities. Almost as important were some of the indirect results of this influence; for although the development at Harvard was independent of the Johns Hopkins movement, it is significant that this was also the period when graduate work in mathematics was placed on a permanent basis at Harvard.

The course of lectures delivered at Johns Hopkins by Cayley in 1882 had fourteen regular attendants, and added materially

to the force of this movement. After Sylvester's departure in 1883 his pupils and associates continued his influence. Cajori reports nearly a dozen institutions in which graduate instruction was offered in the decade 1880-1890. Among them he mentions specifically Harvard, Cornell, Princeton, Virginia, South Carolina, Texas, Michigan, and Wisconsin, as well as Johns Hopkins, and indirectly refers to the work at Yale.

After the departure of Sylvester and Cayley it was natural that the more ambitious students should wish to continue their work under other mathematicians of the same order. Hence we find that by 1885 a number of Americans were studying in Europe, mainly in the German universities. It was no new thing for an American to receive a doctor's degree abroad, but up to this time such cases had been relatively few. From 1885 to 1893, however, American mathematical study continued more and more to derive its inspiration from the seminars of Leipzig, Berlin, and Göttingen. The period of English influence was replaced by one in which a German training was the greatest desideratum, and if a single man is to be designated as Sylvester's successor, it is probably correct to say that the majority of Americans who studied mathematics in European universities received their training at this time in the seminar of Klein. This period was marked by the institution of strong mathematical departments in the newly founded Clark University (1889) and the University of Chicago (1892) and culminated in the Mathematical Congress held at Chicago in 1893 during the World's Fair, and Klein's Evanston lectures.

By 1893 Americans trained abroad had become an influential, and later a controlling element, and from that time American mathematics stood on its own feet. At present doctor's degrees in mathematics obtained each year at home and abroad are about in the ratio of five to one. It is, however, true that many still go to the great universities of Europe, and it will be shown in a later section of this report that at least half of those giving graduate instruction in this country have studied abroad.

III. *Organization and Personnel.*

At present practically all of the leading American universities have organized their graduate work into a separate school or college. With two or three exceptions, however, there is no division of the teaching force into graduate and undergraduate

faculties. In particular, the same department of mathematics gives both advanced and elementary instruction, though not all its members may participate in the former. The number of those who give no undergraduate courses is very small. The work of the department is usually directed by one of the professors, designated as the head or chairman.

An examination of publications of the twenty-four institutions mentioned in Section I shows that the total mathematical teaching force, not including student assistants, was 202 in the year 1908-1909. Of these 136 offered advanced courses, but probably a considerably smaller number actually gave such work, since it will appear in Section IV that the total number of graduate students specializing in mathematics at these institutions was less than 250. It thus seems that only about half the total teaching force at these institutions had a part in graduate instruction. It should be added, however, that in a number of cases large technological departments necessitate a great amount of elementary mathematical work and increase the number of instructors who must devote all their time to such courses. In only two or three institutions was advanced work given by every member of the department.

A further inquiry as to the amount of advanced work given by the 136 instructors mentioned in the preceding paragraph shows that 45 undertook courses announced as primarily for graduates amounting to more than three hours per week. In only five universities were there four or more men offering more graduate work than this, while in ten there was either only one, or else no member of the department who offered more. It thus appears that the greater part of advanced instruction is in the hands of about one fifth of the total teaching force. Naturally these are the older men, as a rule, but it is also evident that a considerable number of others give a limited amount of graduate work. In many institutions each of the younger instructors offers a single graduate course.

In the past the complaint has frequently been heard that American professors are overburdened. That this is still the case, even with men giving a considerable amount of advanced work, is evident from the fact that 9 of the 45 just mentioned have a teaching schedule of from 14 to 16 hours a week. However, there has undoubtedly been an improvement of late in this respect, and the average for all the 45 is about 11 hours per week, a figure that still leaves much to be desired.

As to the preparation of those announcing graduate courses, it appears that nearly five-sixths have obtained the doctor's degree, and probably half of the remainder have had an equivalent training. The instructional force would thus seem to be well equipped. More than half have received training in European universities; a partial list of those who have spent at least one year abroad shows 50 who have studied in Germany, 14 in England, 5 in France, and 1 in Italy.

There is a marked tendency in some of the older universities to recruit their teaching staff from their own students. Even when a period of residence elsewhere has intervened such a practice has some disadvantages, but it also tends to give a permanent character to the work of the department.

IV. *The Graduate Students.*

It may be said generally that the one prerequisite for status as a graduate student in an American university is the possession of a bachelor's degree. In many cases it is announced that this degree must have been given by an institution of equal rank, but as a matter of fact this stipulation is almost never enforced. The result of this is that there is no uniformity of preparation among those classed as graduate students, and frequently such men are taking work as elementary as the first course in the calculus. There are usually, however, more definite requirements to be fulfilled for admission to candidacy for a master's or a doctor's degree, but some institutions make no distinction of this sort and none require all graduate students to qualify as candidates for those degrees. In view of the wide range of attainment which presents itself among those classed as graduate students, statistics on this subject have only a relative value.

Another difficulty that arises is in the estimate of the work of the summer schools which have sprung up in the last fifteen years in connection with most of the larger institutions. Many of these have an attendance of from 500 to 1,500, a considerable proportion being teachers in secondary schools. The instruction is given by regular members of the university, or others specially engaged. The brief duration of these sessions, usually six weeks, and the fact that this is vacation work, very often brings about a serious lowering of standards. Usually, only a small amount of advanced work is offered, and the pub-

lished lists of students give no clue as to the number taking mathematical courses.* For this reason we shall here attempt to give no statistics as to graduate work in summer schools beyond the statement that ten institutions out of the twenty-four appear to give such courses, and that at least in two or three western institutions the enrollment in these classes is larger than the registration in regular advanced work.

The total number of graduate students in attendance on mathematical courses for more than one term or quarter of the year 1908-09 was probably about 400. The 24 institutions especially investigated report 295, but many indicate only the department in which major work is carried, so that a considerable number specializing in physics and astronomy are not included in this figure. It is easier to obtain a fairly reliable estimate of how many have mathematics as their primary subject. The institutions mentioned report 225, not including summer students and those enrolled for only one term or quarter. Possibly 25 more should be added for universities outside this list, making a total of 250, a number less than twice that of instructors offering advanced courses. It may be interesting to note that 40 of the above-mentioned 225 were women. Only two of the leading universities now restrict their graduate attendance to men.

In this connection it would be highly desirable to present some statistics on the growth of graduate work in mathematics and its relation to the growth of graduate study in general. Unfortunately, little is to be found bearing on this point. In the report of the United States Commissioner of Education for the year 1896-1897 (volume 2, page 1649) the number of mathematical students in 24 institutions is given as 247 out of a total graduate enrollment of 3,204, but this probably includes summer registration and students specializing in other subjects. Since that time the number of graduate students in all departments has more than doubled, and mathematics would seem to have nearly kept pace with the general development. In 1904 the same authority (volume 2, page 1424) gives figures for ten

*The summer quarter at the University of Chicago differs considerably from the summer session in other institutions in that it is of equal duration with each of the other three quarters, and is in other respects the same except in the character of the attendance. It is, however, divided into halves, and many students attend for six weeks only. It may be noted that the number of graduate students in mathematics enrolled in this one session is three times the average registration for any other quarter.

institutions, showing 184 graduate students specializing in mathematics, inclusive of about 80 in summer sessions. This total has been increased by nearly 40 per cent in the last five years. It appears, however, that most of this recent growth has been confined to three or four institutions. Columbia reports an increase of over 300 per cent, and there has also been noteworthy development at the University of Illinois, large sums having been appropriated for the graduate school within the last two or three years.

Another indication of the increase of graduate students is afforded by a statement in the BULLETIN of the American Mathematical Society (volume 16, number 2, page 95) in which the number receiving the doctorate in mathematics from American universities in each of the years 1899-1909 is given as, — 13, 11, 18, 8, 7, 14, 21, 11, 13, 22, 15. It may be noted that the average for the first three is nearly six-sevenths that of the last three. Since regulations affecting the doctor's degree are not generally more stringent than they were ten years ago, and no larger percentage of Americans are now studying abroad, these figures naturally lead one to inquire whether advanced work is really increasing as fast as the enrollment would indicate. It is possible that special courses for teachers have largely swelled the totals without materially adding to the amount of research. Another cause for the small and apparently decreasing percentage of doctorates may be the great demand for teachers of mathematics, due largely to the recent growth of technological work; this pressure has probably led many to discontinue their studies earlier than they would otherwise have done. Some light is shed on this point by the fact that in the year 1908-1909 about sixty took the master's degree with mathematics as a major subject. Probably most of these had only that degree in view and terminated their graduate work as soon as it was obtained.

The bulk of advanced work is done by students receiving financial aid, or on salary as subordinate members of the teaching staff. To be more specific, out of the total of 225 in the institutions mentioned, 57 received fellowships and scholarships, paying from \$100 to \$800 per year, and 61 were assistants or instructors. We may safely assume that a considerable proportion of the remaining 107 were secondary teachers or others taking but one course. There would thus seem to be relatively few giving the major part of their time to study without remuneration.

neration, and in fact only four institutions of the twenty-four have as many as six students of this class. The large number of assistants and instructors in graduate classes gives a hint of what is often a most unfortunate state of affairs. These are men who should be devoting all their time to graduate study, but they are usually so loaded down with class-work as to be unable to do justice to the courses they attempt to carry, and so poorly remunerated that they cannot save enough to provide for the additional year or two of research work necessary for the doctorate.

An examination of the migration of graduate students of mathematics develops some very interesting facts. In the institutions mentioned, 70 per cent received their bachelor's degree at another college or university, and about 50 per cent had not received their undergraduate training at any of the other places on that list. One may gather that there is no great disposition among students in universities maintaining advanced work to go elsewhere for further study after receiving the bachelor's degree, but that there is a large flow from the small colleges to the graduate schools. This movement is a source both of strength and weakness to the latter; of strength from the constant accession of fresh and in the main good material; of weakness in that the preparation of many of these men is very defective. Only 25 per cent of the total enrollment had previously taken graduate work elsewhere, from which it appears that there is relatively little migration after graduate status has been attained.

V. *Courses of Instruction.*

Before considering specifically the nature and extent of the courses offered, we will briefly indicate the limitation of their field. Only two institutions have separate departments for applied as distinguished from pure mathematics. Another assigns applied mathematics to the department of astronomy, but it is more usual to include courses of this kind under the head of physics. Frequently the field of this latter department has no clear demarcation from that of mathematics; in general, courses in mechanics and mathematical physics are offered by both departments, with some resulting duplication. This report will cover only the work specifically listed as mathematics.

Courses beyond the first year in calculus are nearly always

announced under two headings, introductory and advanced, often styled "For Undergraduates and Graduates," and "Primarily for Graduates," respectively. Although, as indicated by their title, courses of the former group are attended largely by undergraduates, a considerable proportion of the graduate enrollment is also to be found here, so that they properly form part of the field of the present report.

The American university is apt to be distinctly liberal in its announcements. Thus we find institutions with relatively small equipment offering from fifteen to twenty-five courses in the two groups, and in fact the average for the twenty-four mentioned is about twenty. It should be added that in many cases it is stated that these courses are offered in alternate years, but too frequently the lists are padded and no indication is to be found as to exactly what work was done in a given year. Something less than half the courses announced seem to have been conducted in 1908-1909, about one third of these being of the first group. We may note that the number of courses was thus about the same as that of graduate students, so that the average enrollment per class must have been very small.

The figures that have just been given clearly indicate the ambition of American universities to become centers of graduate study, and raise the question whether in many cases too much is not attempted on too little capital. It would seem that a concentration of advanced work in a dozen or so institutions would be a distinct gain, and that economy in the number of courses might in many cases be advantageously practiced. Often a course is given, not because it is needed, but in order that certain members of the department may participate in graduate instruction. There are, however, several institutions among the twenty-four mentioned which wisely limit their work to a fairly complete list of courses in the first group, with but four or five each year of the more advanced sort. In such cases an adequate training for the master's degree is aimed at, but students who wish to go further are urged to continue their work at one of the larger centers of graduate instruction.

Naturally there is much diversity in the classification of subjects under the two groups; thus we might expect Harvard, which encourages an unusual amount of specialization in the undergraduate program, to include in the first group such courses as quaternions and vector analysis, infinite series, special topics in higher algebra, and theory of numbers. Dif-

ferential geometry is similarly classed at Yale, Michigan, and Wisconsin, as well as one or two other institutions, while about the same number place in this group a first course on the theory of functions.

As already indicated, much of the work in applied mathematics is left to the department of physics. This is particularly true of introductory courses. Only nine institutions of twenty-three whose practice was ascertainable announced under the head of mathematics a first course in analytic mechanics. Eight offered a second course in mechanics under the first group, but in only five cases was this given. It is, perhaps, more to be expected that celestial mechanics should be offered as a mathematical subject in but two institutions.

There is substantial agreement in classing the following subjects under the first group: advanced calculus, advanced analytic or projective geometry, solid analytic geometry, determinants and the theory of equations, and an introduction to differential equations. The first two are almost always full year courses while the others are more often of half that length. The extent to which these may be considered standard subjects is indicated by the fact that but one institution on the list fails to offer a course in advanced calculus, and in only one other instance was such a course not given, if we accept the evidence of catalogues; and practically the same is true of the second named subject. Three failed to offer the third and the same number the fourth as separate courses in the group, though we may suppose the subjects to have been included elsewhere, while the last was everywhere listed, either by itself, or in a few cases as the second half of the course in advanced calculus; and in only a few additional cases were these courses not actually given.

Though there is thus seen to be a considerable uniformity as to introductory subjects, it is interesting to note some of the variations from the more usual lists. Five courses in the history of mathematics are announced, three in probabilities and one in finite differences, six in statistics and insurance, and one in the theory of envelopes. Harvard and Iowa combine algebra with projective geometry, and as noted above, in some cases the first course in differential equations is given as a part of advanced calculus. There would thus seem to be evidence of a tendency to consolidate the introductory work into as compact a group as possible, and this tendency is probably on the increase.

The establishment of special honors courses similar to those of Oxford and Cambridge has been much discussed of late, and has been recommended in recent reports of several university presidents, but so far Princeton and Columbia alone have adopted such a system. This movement, though of great interest, will not be further described here as it at present concerns undergraduates only.

In the consideration of the group of advanced courses it will be useful for some purposes to introduce what may appear to be an arbitrary distinction in classing institutions; those which gave more than six courses of this kind will be placed together, while the others will be regarded as constituting another division. As a matter of fact, however, this corresponds to a real difference in the character of the work undertaken; the thirteen universities of the former class all have candidates for the doctor's degree among their students, while this is true of almost none of the others. As might be expected, a majority of the institutions giving more than six advanced courses are to be found among the older universities of the east.

As to subject matter, we may note that nearly everywhere far more emphasis is placed upon analysis than upon algebra, geometry, or applied mathematics. In but one institution of the first division are more courses given in any one of the last three subjects than in analysis, and the same is true of the class enrollment. As to the relative attention paid to the other branches, this varies widely, depending chiefly on the special interests of members of the department. Only two institutions of the first division leave all work in applied mathematics to other departments, and only two fail to offer courses in both geometry and algebra. We may conclude that there is generally a conscious effort to give a balanced curriculum, with especial emphasis on such subjects as the teaching staff is best prepared to offer. With institutions of the second division an even distribution of each year's work is, of course, impossible. Most of them, however, vary the courses given from year to year so as to include as wide a field as possible. In a few cases this seems to be only a matter of intention, and the courses actually conducted are confined to the subjects most interesting to the two or three men who have charge of graduate instruction.

All the universities of the first division include in their program from two to seven research courses or seminars, while

none of the others has more than one. As this matter lies in the field of another committee, it will not be further discussed in this report:

Both in institutions of larger and of smaller equipment the practice of varying the program from year to year prevails. This is obviously a necessity where the enrollment is small. A number of the universities of the second division make it a rule never to repeat an advanced course in successive years, and this seems also to be true of at least three of the first group. In only two or three institutions are there more than four courses repeated year after year, and even in these cases the subject matter is seldom exactly the same. By far the most usual method of announcing courses consists in grouping them in pairs, sometimes from the same field but often widely differing in nature, offered by the same instructor in alternate years. Several institutions announce all of their work in this way. In a few cases more extended sequences are offered in which the cycle consists of more than two years. The effort thus shown to give variety to the work of the teaching staff and to cover as broad a field as possible is in the main most commendable, but it sometimes leads to the giving of advanced courses by men who may be qualified to do work in certain lines but are obviously unprepared for what they undertake.

Although there is a marked tendency not to devote whole courses to special topics, and one may note a certain conservatism in choice of subjects, most institutions offer at least one course in some comparatively narrow field, frequently one of recent development. Thus we find integral equations (given by three institutions) as well as point sets, divergent series, functional equations, modern theories in geometry, general analysis, differential invariants, linear systems of algebraic curves, imaginaries in geometry, symmetric functions, elasticity, and capillarity. Advanced courses in insurance, probability, and statistics appear in three announcements. In two instances the influence of Sylvester probably accounts for such subjects as the logic of mathematics, and principles of mathematics.

Although this may encroach somewhat on the field of another committee, we may at least mention here the courses offered specifically on the teaching of mathematics. As might be expected, these differ widely in nature. We shall not here enter into their character, but merely report that all but seven institutions of the list offer work of this kind.

This section should not be concluded without some reference to the courses of lectures delivered by distinguished foreign mathematicians on various occasions within the last thirty years. It is true that comparatively few graduate students heard them, but the influence of Cayley's course has already been indicated, and the visit of Klein in 1893 was probably no less important. Of late a number of such courses have been given, and the exchange system may possibly lead to further developments along this line.

VI. *Methods of Instruction.*

Practically all the work, both introductory and advanced, considered in the preceding section, is given by means of lectures of a little less than an hour in length. Sometimes seminars or research courses meet for two consecutive hours. By far the most usual program consists of three lectures a week, but three or four institutions prefer two-hour courses; very few classes meet but once a week, or oftener than three times.

A questionnaire on methods of instruction was sent to eighteen of the twenty-four institutions previously mentioned and replies were received from the following: Bryn Mawr, California, Chicago, Columbia, Cornell, Harvard, Illinois, Michigan, Missouri, Northwestern, Pennsylvania, Princeton, Wisconsin, and Yale. It appears that on some points there is a considerable diversity of practice, even among instructors in the same institutions, for freedom of instruction is, fortunately, almost everywhere carefully preserved. However, in most cases the answers received stated that the department used substantially uniform methods.

The first question concerned the assigning of courses to the members of the department, and was put in the form, — Are standard courses (the theory of functions, for example) given by the same or by different instructors each year? Only two answers indicated that the latter alternative was the rule. Two more, however, stated that a change in this direction was probable, while four indicated some variation in practice. The custom of permanently assigning certain fields to certain men thus seems to be distinctly in favor. In spite of the advantages of such a system it is undoubtedly true that in many cases it is carried too far, so that a subject comes to be considered a particular instructor's preserve, not to be trespassed upon.

An inquiry into the matter of examinations reveals two

widely differing conceptions of graduate work. On the one hand the practice of Harvard, for instance, in requiring searching written examinations at the end of each semester indicates a general system of careful supervision, while Columbia, Princeton, Chicago, Bryn Mawr, and California, which, it seems, ordinarily give no course examinations, represent the opposite extreme where the student may take or leave the intellectual repast spread before him. Two or three other institutions allow oral examinations, and in some cases there is a difference of rigidity in the conduct of introductory and advanced courses. It appears that Yale, Harvard, Wisconsin, Northwestern, and Missouri represent the maximum of strictness in requiring written semester examinations. It is hard to determine any tendencies in this matter.

Another question that has a similar bearing concerns the giving of grades indicating excellence of work, a custom which prevails in undergraduate courses in American universities. The answers received show that the opposite practice is most usual in graduate work, but Harvard, Wisconsin, Chicago, Illinois, and Northwestern report that such grades are given.

To the query, — Are problems assigned for report from time to time? — an almost universal affirmative was returned, so that it would seem incorrect to conclude that anywhere is the student expected to be merely a hearer of lectures. It appears that in some courses in Columbia and Chicago this may have been the case, but the universal rule elsewhere is to require reports of some kind on assigned work. On the other hand stipulated reading in connection with courses is almost never insisted on. Lists of authorities and references are assigned but the responsibility for consulting these is thrown upon the student. Only three replies indicated an occasional exception to this rule, while but one or two stated that reports are required on such outside reading.

As already said, the work is given almost entirely in lectures, though only five replies indicated that text-books were never used. However, it would seem that they are followed in only a few introductory courses, and that even here they are used rather as reference books. Perhaps the most usual exception is in the first course in differential equations. The replies received were unanimous in stating that there were no courses in which a text-book was used exclusively, no lectures being given.

It is customary to devote some hours, or parts of hours, to reports of students and discussion of lectures. However, at Harvard reports are given in seminars only, while at Princeton and Northwestern they are very infrequent. Yale, California, and Cornell do not give class time to discussion of lectures, though the last named appoints regular hours for individual consultation. At Wisconsin oral quizzes on lectures are a feature. The most usual practice everywhere is to allow a few minutes of each hour for discussion, questions being frequently asked in interruption of a lecture. The instructor's procedure is as a rule quite informal. In some cases an hour is reserved once a week or twice a month for discussion and reports.

In addition to the regular lecture and seminar courses, some institutions maintain courses for reading and research, in which the student (usually there is but one) meets the instructor at intervals for report and consultation, but no stated lectures are given. Often; but not always, this is done in connection with the writing of a thesis. Harvard announced eight such courses last year, Cornell two, Michigan two, and Missouri one. In the last two cases only were written examinations given, but in all but two courses problems were assigned for report.

Work in absentia has decreased almost to the vanishing point, Chicago alone conducting a few graduate courses by correspondence.

A number of the larger institutions are well supplied with models, calculating machines, and other mathematical apparatus, but others are more meagerly equipped. In several cases the making of models has had a place in graduate instruction. A special feature at Yale has been the writing up of lecture notes by students and placing them for reference in the seminar room. This practice has been followed elsewhere for certain courses, but does not seem to be the rule in any other institution.

In the main it would appear that little change in methods is contemplated, and certainly no general movement is on foot. Two or three institutions have inaugurated considerable departures from past usage, but this is in each case an individual matter due to changes in the personnel of the department.

VII. *Requirements for the Master's Degree.*

The master's degree "in course" is given by practically all American colleges and universities as representing one full

year of graduate study.* The title of Master of Arts or of Science is also given as an honorary distinction under entirely different conditions, but we will here consider only the degree "in course." This is sought especially by students who either have but one year to devote to advanced work or are not sure of being able to continue for the doctorate, since its possession is often a decided advantage in obtaining a teaching position. Even in secondary work the master's degree is frequently insisted on, and this tendency is so marked that in the opinion of many it will before long be as much in demand there as the doctor's degree now is for the candidate for a college position.

Excluding varieties of nomenclature corresponding to technical or professional courses, four master's degrees are given, namely, Master of Arts, Master of Science, Master of Philosophy, and Master of Letters, but of the last two the former is conferred by but three institutions of the twenty-four previously referred to, and the latter by only one. Nearly half give the master's degree in arts only. The distinction is, except in a few instances, an artificial one in that it does not indicate the candidate's field of advanced work, but merely shows which bachelor's degree he has received. It is noteworthy that in eleven institutions of the twenty-four this distinction has been abandoned, and even where the degree in science is preserved it is often possible for any who prefer it to receive the arts degree provided their work is not mainly professional or technical.

As noted on page 236, about sixty master's degrees were conferred on students specializing in mathematics in the year 1908-1909, a number but little less than one-fourth the total graduate enrollment in that subject. Of the twenty-four institutions especially considered, only five or six failed to confer the degree in that year. Two institutions, Harvard and Columbia, gave the maximum number, eight each.

The universal condition for admission to candidacy is the possession of a bachelor's degree from an institution of equivalent standing. Although there is some latitude in the interpretation of this rule, it is but fair to say that the student admitted with inferior preparation is usually required to make up his

* Johns Hopkins University seems to be alone in requiring two years. Bryn Mawr gives the degree to no students who have done their undergraduate work elsewhere, except as an accompaniment to the doctor's degree.

deficiencies, taking a longer period than the minimum of one year for obtaining the degree. Applications for candidacy accompanied by a program of studies must usually be filed early in the first semester and must be approved either by a committee of the faculty, or by the dean of the graduate school. In case a thesis is required, the subject must be submitted soon after admission to candidacy. The number of courses which must be carried varies from three to five, but in the latter case an allowance is made for the thesis. Some of this work may be done in summer school, and in a number of institutions the degree may be obtained by attendance on three or four such sessions, with the usual requirements as to examination and thesis. Correspondence study or work in absentia has nearly disappeared so far as mathematics is concerned, though some catalogues still announce that part or all of the requirements may be thus met by students who have previously been in attendance. In a few cases where five three-hour courses must be carried a deduction is made for advanced work taken by the student as an undergraduate, but in excess of the requirements for the bachelor's degree. A combination with professional work is sometimes possible, but in the case of mathematics this must be confined chiefly to teaching and engineering.

In 1904 a committee, after two years' consideration and careful inquiry, reported to the Chicago Section of the American Mathematical Society on the requirements for the master's degree, with especial reference to conditions in western institutions. We will here use their recommendations, which, they state, correspond with the requirements at several leading American universities, as a basis for discussing present conditions. Their findings will, however, be only briefly summarized, since the details are easily accessible.*

The first questions raised are, — What should be the minimum acceptable time devoted to undergraduate mathematics, and how should it be distributed? — The committee agrees that this should be the equivalent of five three-hour courses carried continuously throughout the four years of the undergraduate course, and distributed about as follows: In preparation for the calculus (algebra, trigonometry, and analytic geometry), five year-hours; calculus, including differential equations and mechanics, six hours; advanced algebra and geometry, four hours. It will be noted that this list includes

* Published in the BULLETIN, 2d ser., vol. 10, No. 8 (1904), pp. 380-385.

seven year-hours in courses classed as "introductory" in section V. Courses in applied mathematics are not specifically recommended, though it is urged that undergraduate instruction maintain a close contact with the physical sciences. Except that four undergraduate courses are more usually taken as a minimum, and that the exact distribution here shown is not always insisted on, it may fairly be said that the stronger institutions at present comply very closely with the above conditions.

The next section of the report reads: "Upon the basis of the preparation indicated above, it is the opinion of the committee that the work leading to the master's degree should include at least 270 hours * selected from the following groups of subjects. The selection should in any case include subjects from at least two of the three groups.

I. GEOMETRY.

Projective geometry.

Modern analytic geometry, algebraic curves and surfaces.

Application of calculus to twisted curves and surfaces.

(Differential geometry.)

Solid analytic geometry.

Descriptive geometry, (Darstellende Geometrie.)

II. ANALYSIS.

Theory of equations.

Advanced calculus.

Theory of functions.

Differential equations.

Theory of numbers.

Invariants.

III. APPLIED MATHEMATICS.

Analytic mechanics.

Mathematical astronomy.

Mathematical physics.

Mathematical theory of probability.

"This arrangement provides that the candidate, in case he enters upon his graduate work with the minimum mathematical preparation indicated above, and hence having a broader *general* training in his undergraduate course, should be required to take at least two thirds of his year's graduate work in pure and

* The term hour here refers to a single lecture hour, so that the above is equivalent to two and one half three-hour year courses.

applied mathematics. On the other hand, as it may be assumed that the student entering upon his graduate work with a larger mathematical credit than this has had less *general* training in his undergraduate course, it is the opinion of the committee that he should be permitted to take as much as one half of his work in departments other than mathematical, providing, however, that the minimum requirement in mathematics indicated above is also fulfilled."

Present practice tends to reduce somewhat the minimum here recommended, so that in a majority of the larger universities two courses in mathematics are accepted, though as a matter of fact more are usually taken. Four institutions require a candidate to present three subjects and two others require work in at least one additional department. The majority would seem to permit the student to take all his work in mathematics if he so chooses, but with one or two exceptions there seems to be no institution where this is distinctly encouraged. No general rule obtains as to the distribution of the candidate's work; this is nearly always an individual matter decided by the instructor or committee in charge. It is probable, however, that the recommendation presented above represents nearly the average of practice.

An interesting variation from ordinary requirements is presented by the University of Michigan. In addition to "Plan A," which calls for the usual amount of specialization, a scheme styled "Plan B" allows a wider choice of studies. Here the thirteen hours to be taken may be distributed as is seen fit among three departments and other requirements would seem to be less rigid, but the year's work will not usually count as one of the two or three necessary for the doctor's degree.

It is apparently nowhere stipulated, except at Bryn Mawr, where unusual conditions obtain (see footnote, page 245); that the candidate be able to read a foreign language. The report under consideration recommends a knowledge of German and considers French also desirable. As a matter of fact the requirements for the bachelor's degree are such that nearly every graduate has had at least one year's work in one of these languages, but seldom enough to make reading easy. Nevertheless, in most cases some reading of foreign mathematical texts is urged, though rarely required.

The committee quoted believe that all candidates should present a thesis, but that this need not embody original research.

It appears that thirteen institutions of the twenty-four adhere strictly to this rule, while in at least five more the requirement is optional with the instructor in charge or the head of the department. In three or four others a thesis is never presented for the master's degree.

The last topic considered is that of examinations. The report mentioned recommends either a final test covering all the graduate courses taken by the student, or the requirement that a high grade be obtained in term examinations in each course. The former alternative is the more usual one, — in fact it obtains at eighteen of the twenty-four institutions considered, while in several of the others either procedure is possible. At Harvard one of the two highest of the four passing grades in each course is the sole requirement. The final examination may be either oral or written, — practice seems to be fairly evenly divided in this matter, — and if oral, it is usually conducted by a committee of instructors from the departments involved.

There are smaller and less reputable institutions in which the requirements are lower, and in fact some where all requirements are merely nominal. The present report has aimed only to present the state of affairs in the institutions which are really prepared to confer the degree.

GEOMETRIC TRANSFORMATIONS.

Geometrische Transformationen. Zweiter Teil: die quadratischen und höheren birationalen Punkttransformationen. By KARL DOEHLEMANN, associate professor of mathematics at the University of Munich. Leipzig, Göschen (Sammlung Schubert, XXVIII), 1908. viii + 328 pages and 59 figures.*

AT the time this volume was written there was no book devoted to the subject of its title. The next year appeared the fourth volume of Professor Sturm's treatise † and several parts of the *Encyklopädie*, so that now we are fairly well supplied with an introduction to this most fundamental study. And the two books do not overlap; the larger work is synthetic and replete with geometric illustrations, while that under review is

* The first part of Professor Doehlemann's treatise appeared in 1902; it was reviewed in the *BULLETIN* by Professor Gale (vol. 10, pp. 512-515).

† See the review in the *BULLETIN*, volume 17 (1910), pp. 150-154.

analytic, elementary, and well adapted to its purpose of serving as a finder into the literature and problems of birational transformations. Perhaps it would have been well to have the name Cremona transformations in the title, for no mention is made of the geometry on a given curve or surface, hence such concepts as point groups, series, residuation, adjoint curves, etc., are entirely absent, as is also the entire theory of multiple correspondences and higher involutions. The selection of a restricted field is a wise one; to treat adequately the subject of birational transformations in this broader sense requires several volumes, and goes beyond the task imposed in the Schubert collection. In fact it is the introduction that is particularly needed. If a reader has a thorough knowledge of Cremona transformations, he is well prepared to study the geometry on a curve or surface by the Brill-Noether method.

The book is divided into two nearly equal parts, the first treating of the plane, the second of space. The first chapter introduces quadratic transformations by means of two pairs of projective pencils of lines, first synthetically, then algebraically. The transformation of a given curve is discussed for a large number of special cases, including the images of an extensive list of singular points. This part is developed synthetically only, thus not furnishing the reader with complete weapons for the resolution of the singularities of a curve whose equation is given. The general bilinear equations are now shown to define the same transformations; the cases in which two or three of the vertices of the fundamental triangle approach coincidence being briefly considered. Finally, the case where the fundamental triangles of the two planes coincide and the condition for involution are discussed. At the end of this and of each subsequent chapter is a list of the memoirs bearing on the topics considered. These lists are not complete, and a few conspicuous absences are noticed, but on the whole they are well selected and give a very good idea of the literature and of the unsolved problems.

Now follow two chapters (pages 50-134) on metrical properties based upon the case in which two of the vertices of the fundamental triangle are the circular points at infinity. Foci, minimum lines, reciprocal radii, orthogonal circles, and analagmatic curves are discussed in a very elementary manner, most concepts being developed independently, then shown to belong under the preceding projective treatment. A number of applications to mechanics, inverting straight lines into circles, are

considered, and a number of properties of bicircular quartics and of circular cubics are given. The second of these chapters considers the bilinear relation between two complex variables, the development being similar to that given in most books on the theory of functions. It includes conformal mapping and gives a number of elementary examples. While it is interesting to note the position of this transformation as a particular case of those mentioned before, it seems hardly justifiable to devote over one tenth of a book concerned with geometric transformations to the discussion of a case whose main use is in analytic problems, and which has been treated repeatedly elsewhere. This, however, is the only case in which the perspective of relative importance (for geometric purposes) does not seem to be preserved in the best way. Even here it might be said that all the steps of the general case can be understood most easily by a careful study of the simple one.

Chapter IV, on the general birational transformation of the plane, commences from an entirely different point of view. If

$$x'_i = \phi_i(x_1, x_2, x_3)$$

defines a rational function of the x_i , under what conditions can the equations be solved rationally for x in terms of the x'_i ? In the search for the answer to this problem we naturally find the Cremona net, meet with fundamental curves and fundamental points, and discover a number of relations existing among them. The Jacobian of the Cremona net is shown to consist of all the fundamental curves of the transformation. A table of all the forms up to $n = 10$ is given, n being the order of the rational curve, image of a straight line. The theorem that every Cremona transformation can be resolved into quadratic inversions is proved in detail when the basis points are distinct; mention is made of the modifications necessary when two or more of these approach coincidence, and the memoirs treating of these cases are cited.

The first part closes with a rapid glance over the newer problems, most of the theorems being given without proof. Since the name of the author appears frequently in the literature of these sections, he is particularly well fitted to prepare such a survey. The involutorial transformations are discussed in some detail, in particular the Jonquières transformations. The results of the researches of Bertini regarding the number of types of irreducible involutions are given, and a method of

obtaining all except that of order 17 is added. A section on periodic transformations gives a résumé of the results of Kantor, discusses conjugate and commutative transformations but does not give any geometric criteria. The concept of finite groups of Cremona transformations makes still further demands on the algebra of substitution groups; the results obtained by Wiman are given without proof. A short statement concerning continuous groups is also made, principally to refer to the literature. Whether there are curves which remain invariant under an infinite discontinuous group is not mentioned.

In the second part, the quadratic transformations of space are introduced by means of two reciprocal systems. The treatment is synthetic, and includes a discussion of fundamental elements and of the image of a general curve and a general surface. Then follows an algebraic development of the same idea, the two together making a symmetric and well balanced treatment. Particular cases are discussed briefly but not exhaustively. One misses the inverse of the general quadratic transformation. It is introduced later, but here the reader meets the sole statement that the inverse of a transformation may be of different order than that of the direct operation. The generalized stereographic projection and the statement that the image of a plane is a surface which can be rationally mapped upon a plane fittingly close the chapter.

Now follow two chapters (pages 201-286) on the particular case in which the fundamental conic is the absolute circle at infinity. As in the plane, the discussion is very elementary and proceeds algebraically with non-homogeneous coordinates. Besides the geometry of reciprocal radii a number of applications are made, particularly to the simple stereographic projection and Mercator's chart. Then follows a systematic geometry of the sphere, including complexes, congruences, depiction into hyperspace, and the generation of Dupin's cyclides. Thus far everything is linear; the discussion of the quadratic equation is much more brief. No mention is made of the higher geometry of the sphere, which contains perhaps the most elegant example of birational transformation in the whole field. The excuse may be made that in the elementary orthogonal case we are still dealing with a point transformation, whereas in the general case we are not. On the other hand the entire treatment is in terms of the sphere as element of space, and its coordinates are interpreted as point coordinates in hyperspace.

No apology need be made for this long digression into the geometry of the sphere. Here the purpose is a geometric one and the discussion keeps well within the limits set by the book. The surfaces which naturally appear in this way can be interpreted projectively. If, finally, the line-sphere transformation be applied, we have the Kummer surface and as particular cases the Steiner surface and the quartic ruled surfaces. Together they comprise practically all the quartic surfaces that we know anything about. A short section on electric images contains a list of most of the important memoirs devoted to it.

Chapter VIII, on cubic and higher birational transformations of space, begins with three bilinear equations analogous to the pair previously treated in the plane. It is exceptionally well written and contains all the important principles for the discussion of the $(3, 3)$ transformation, that is, that in which planes go into cubic surfaces, and the inverse is of the same form. By means of these principles it is easy and natural to discuss the mapping of a general cubic surface on the plane. In the treatise of Sturm this depiction precedes the consideration of the cubic transformation, but the treatment of our author seems more natural and is much easier to follow. The details are not given, but three different depictions are outlined.

The last section contains a brief treatment of the general birational transformation, including a discussion of the fundamental points, curves, and surfaces. The monoidal case being given as an example, a sketch is added regarding the method of procedure in the determination of new transformations, and in the corresponding depiction of a surface upon a plane.

Of the few typographical errors noted the only ones that may cause confusion are found on pages 140 and 141. The letter i is there used as symbol of summation in two different senses, the first running over the three coordinates, and the other over the h fundamental points of the plane. In the resulting formulas these two symbols are curiously combined.

In the book under review Professor Doehlemann has admirably fulfilled the purpose aimed at in the *Sammlung Schubert*.

VIRGIL SNYDER.

SHORTER NOTICES.

The Collected Mathematical Papers of James Joseph Sylvester.
(Edited by H. F. BAKER.) Volume III. xv + 688 pp.
Cambridge, University Press, 1909.

THE first two volumes of Sylvester's works appeared in 1904 and 1908, and were reviewed by the present writer in the BULLETIN, volume 15 (1909), pages 232-239. The third volume contains chiefly the papers written while Sylvester was at the Johns Hopkins University. The majority of these papers relate to the theory of invariants, in particular with the use of generating functions to enumerate the complete system of concomitants of a system of quantics. In two of the earlier papers of this series Sylvester proved the dependence of two of the thirty fundamental forms of two binary quartics given by Gordan, and emphasized the importance of the English enumerative method in questions on the dependence of the invariants. Among the longer papers is that "On an application of the new atomic theory to the graphical representation of the invariants and covariants of binary quantics." On the origin of this paper on chemistry and algebra, Sylvester says: "Casting about, as I lay awake in bed one night, to discover some means of conveying an intelligible conception of the objects of modern algebra to a mixed society, mainly composed of physicists, chemists, and biologists, interspersed only with a few mathematicians, to whom I stood engaged to give some account of my recent researches in this subject of my predilection, and impressed as I had long been with a feeling of affinity if not identity of object between the inquiry into compound radicals and the search for 'Grundformen' or irreducible invariants, I was agreeably surprised to find, of a sudden, distinctly pictured on my mental retina a chemico-graphical image serving to embody and illustrate the relations of these derived algebraical forms to their primitives and to each other which would perfectly accomplish the object I had in view."

Mention should be made of the elaborate memoir on ternary cubic-form equations in regard to their solvability in rational numbers, in particular the cases in which an integer can be expressed as a sum of two cubes. The paper also treats of the arithmetical properties of cyclotomic functions and of polygons which can be inscribed and escribed about a cubic. The volume contains various short papers on the theory of numbers, in

particular one on the totality of primes comprised within given limits.

The memoir on subinvariants and perpetuants belongs to this period. At the end of the volume appear some short papers on the theory of partitions; but Sylvester's chief work in this field will appear in the fourth (final) volume of his collected papers.

L. E. DICKSON.

Niedere Zahlentheorie. Zweiter Teil. By DR. P. BACHMANN.

B. G. Teubner's Sammlung von Lehrbüchern auf dem Gebiete der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. Band X, 2. Leipzig, 1910. x + 480 pp.

THE first volume of this work appeared in 1902 and was reviewed by the writer in the BULLETIN, volume 9 (1903), page 555. The present second volume has the subtitle Additive Zahlentheorie, a term suggested by Kronecker for the properties of numbers relating to their additive combinations. The first chapter deals with arithmetical series of the n th order, polygonal and figurate numbers, the sum of the k th powers of the first n integers, the sum of the k th powers of n numbers in arithmetical progression, and at length with the theory of Bernoullian numbers. The second chapter deals with recurring series, in particular with those bearing the names Farey, Fibonacci, Fermat, Pell, and Dupré. Application is made to factorization of numbers of certain forms, perfect numbers, Fermat and Mersenne numbers. Here results are omitted that have been known for several years. For instance the number $2^{67} - 1$ is left in doubt although Cole has given two factors (BULLETIN, 1903, page 137). Chapters III-V relate to the theory of partitions. Chapter VI is entitled recursion formulæ and deals with various number theoretic functions. Chapters VII-VIII treat of the representation of a given number as a sum of like powers or as a sum of multiples of powers. Every positive integer can be expressed as the sum of four squares (but not always as the sum of three); as the sum of nine cubes (but not always as the sum of eight); as the sum of 37 fourth powers (though doubtless this limit is too high). Further theorems relate to the number of representations of an integer as the sum of four squares. The related investigations by Liouville are given at length. The final

chapter deals with Fermat's equation $x^n + y^n = z^n$ from the standpoint of the elementary theory of numbers, giving Fermat's proof for $n = 4$, the Euler-Legendre proof for $n = 3$, and remarks on the Dirichlet-Legendre proof for $n = 5$, and the Lamé-Lebesgue proof for $n = 7$. Kummer's method by ideals is beyond the scope of the work; the comment on regular primes (page 461) is corrected at the end of the book. The formulas obtained independently by Abel and Legendre are established. The developments by Sophie Germain, E. Wendt, and L. E. Dickson are then cited. In one instance (page 475), the initials of the last name are given incorrectly; while, in a quotation from Sylvester on page 104, permeating is spelled wrong. However, the book is especially free of errata and the typography is excellent. In the present text Bachmann has fully maintained his reputation as to clearness, thoroughness, and exhaustiveness.

L. E. DICKSON.

Eléments de Calcul vectoriel avec de nombreuses Applications à la Géométrie, à la Mécanique et à la Physique mathématique. Par C. BURALI-FORTI et R. MARCOLONGO. Traduit de l'italien par S. LATTÈS. Paris, A. Hermann et Fils, 1910. vii + 229 pp.

So lengthy a review * was recently accorded to two new books on vector analysis by Burali-Forti and Marcolongo that nothing more than the mere mention of the French edition of the first of the two would be needed, were it not for the fact that in the French the authors have added a long and excellent appendix on Grassmann's geometric forms and on Hamilton's quaternions. The object of the appendix is to show the power of the authors' vector analysis by using it to set up the Grassmannian and Hamiltonian systems. There is apparently the further object to set forth these two mathematical disciplines in such a way that mathematicians in general, and in particular those mathematicians who think they know something about the systems, shall be led to conceive or reconceive, as the case may be, these systems as they should be conceived. We have no exceptions to take to the authors' presentation of the subject; it is compelling.

There is one remark, found on page 201, which deserves

* Under the title "The unification of vectorial notations," BULLETIN, vol. 16, pp. 415-436 (May, 1910).

particular attention. The authors have defined quaternions in their way and have given Hamilton's definition and then remark that they will prove that the quaternions given by their definition are the quaternions of Hamilton and not any other kind of quaternions. The importance of this remark is the implication that there are other quaternions than Hamilton's. Of course the authors are not interested in any others any more than they seem to be interested in Gibbs's algebraic point of view; but we are glad to have them imply that there are others — for in the strictest sense it is precisely other quaternions than Hamilton's original ones that most quaternionists now use. Gibbs* pointed out one defection from the strict Hamiltonian point of view; the authors have frequently pointed out others for the purposes of berating the quaternionists who use them.

It may interest some, who had not the opportunity to hear Gibbs, to see (as well as I can remember) the way in which he presented his, not Hamilton's, quaternions. He considered ordinary vectors $\alpha, \beta, \dots$ between which he had already defined his scalar product $\alpha \cdot \beta$, his vector product $\alpha \times \beta$, and his dyad product $\alpha\beta$; and he proceeded to define his quaternion product as †

$$\alpha \circ \beta = -\alpha \cdot \beta + \alpha \times \beta.$$

Thus his quaternion was the sum of a vector and a scalar — which Hamilton's quaternion primarily was not. In particular, scalars and vectors were merely especially simple quaternions. The defining equation gave at once

$$i \circ i = j \circ j = k \circ k = i \circ j \circ k = -1$$

and all the formal laws of quaternion algebra. Here i, j, k were vectors and the product was quaternionic. In Hamilton's analysis the i, j, k would be not vectors but right quaternions, and for him the right quaternion was not primarily a vector. So far as we can judge, the authors are entirely right about Hamilton, and the followers of Hamilton are a good deal wrong about him. They do not any longer use the original quaternions; they use their own, which are practically identical with Gibbs's; it is a pity the authors have not analyzed these that are used instead of the original ones which Hamilton analyzed with such detail.

E. B. WILSON.

* The Scientific Papers of J. Willard Gibbs, vol 2, p. 172.

† He did not use the sign $\circ$ or any other special sign for the quaternion product.

Gaston Darboux. Biographie, Bibliographie analytique des Ecrits. Par ERNEST LEBON. Paris, Gauthier-Villars, 1910. viii + 72 pp.

THIS is the second volume of the series *Savants du Jour*, which M. Lebon is bringing out. It is as regards content and arrangement similar to its predecessor, which was devoted to the life and work of Poincaré.* It begins with a short biography of Darboux (7 pages). A few extracts may be of interest to the readers of the BULLETIN. Darboux was born at Nîmes, August 13, 1842, the elder of two sons of a grocer. His father died in 1849. But his mother, noting the intellectual gifts of her two sons, provided for their education. In 1861 the youthful Gaston was awarded the rare distinction of being admitted as the first among all candidates both to the Ecole Polytechnique and to the Ecole Normale. He chose the latter. He received his doctor's degree from the University of Paris in 1866. After several years spent in secondary teaching, during which period he published several important papers, he became in 1872 "maître de conférences" in the Ecole Normale. In 1875, at the age of 32 years, he was awarded the Prix Poncelet by the French Academy for the ensemble of his mathematical work. Since 1880 he has occupied the chair of geometry at the Sorbonne, being the successor in this chair of Chasles. He was elected a member of the Academy of Sciences in 1884, and made a permanent secretary of this body in 1900. He is a member of 21 academies and of 11 scientific societies, and is an honorary doctor of the Universities of Cambridge, Christiania, and Heidelberg.

The remainder of the little volume is devoted to an appreciation of the scientific work of Darboux taken from various sources, and a complete list of his writings. The latter numbers 419 titles, many of which are accompanied by brief abstracts. The amount and quality of such productivity is at once the inspiration and the despair of his would-be disciples. The volume also contains a fine portrait of Darboux.

J. W. YOUNG.

Handbuch des mathematischen Unterrichts. Von W. KILLING und H. HÖRSTADT. Band I. Teubner, Leipzig und Berlin, 1910. viii + 442 pp.

NOTWITHSTANDING the great number of handbooks on mathematical instruction the authors justify the present work by the

* Cf. review in the present volume of the BULLETIN, p. 42.

claim that such books in general are too abstract or are too much concerned with methods of teaching and do not embody a sufficiently thorough and painstaking investigation of those things which we try to teach. Hence, assuming that the average person who proposes to teach elementary mathematics has neither the leisure nor means to investigate the advance of the science from original sources, the authors devote at least half of the book to presenting the facts in available form for the novice. The aim is obviously to rid the mind of the teacher of the idea that Euclid is sacred, and to fill him with the modern spirit.

It is worth noting that there are practically no drawings in this volume, devoted to geometry, but the required figures are very carefully described and relations of position fully defined. The reason is obvious, for a single figure is of necessity special, whereas if the reader draws a figure from the description, he will find the latter fits any case.

The book naturally divides itself into two parts. The first half is devoted to a careful and well balanced discussion of the foundations of geometry. The authors define a complete system of axioms and proceed to investigate those of Euclid and Hilbert, the latter being assumed to be the best of modern systems. There is a most interesting general discussion of euclidean and non-euclidean geometries, showing how the latter were forced upon certain investigators against their will. The bibliography here as throughout the book is excellent and is one of its most valuable features. The distinctive characteristics of the various non-euclidean geometries are clearly presented.

In order to show the basal ideas in geometry there is an attractive discussion of natural and non-natural geometries. The futility of *defining* a line as "the locus generated by a moving point," and of similar proceedings is pointed out with some degree of humor. The ideas of right and left on the line and plane, of congruence and symmetry, of the equality of area or content in plane figures, and corresponding ideas in space are treated thoroughly. The chapter on non-real configurations of the plane contains some excellent hints for the teacher presenting this subject. The authors express the opinion that most elementary teachers are unable to comprehend and explain the exact nature of the so-called infinitely distant point. Such teachers would do well to read this vigorous treatment. The chapter on plane construction problems is no less inter-

esting. The problem of determining whether a construction is possible by means of a straight edge and compass is naturally placed at the head of the chapter. Numerous constructions are proposed, including a list of apparently simple ones which are however impossible. Constructions possible by means of other instruments are discussed also, and their relation to the first problem explained.

The two chapters on the logic of geometry and on literary form in mathematical instruction are interesting. By a complete system of axioms, we mean one that contains all specifically geometrical ideas, and we fail to notice the axioms of logic for combining them, which we nevertheless tacitly assume. The authors attempt to build up a system of axioms of this logic. The teacher has further a right to demand good literary style as well as logical precision, but the authors deride hair splitting niceties and cranky insistence on logical and verbal completeness, which only serve to kill the student's interest without clarifying his ideas. Clinch the argument by conversation about the theorem, is their advice. The final chapter of the first half is devoted to a rational discussion of the teaching of elementary geometry. The absolute folly of having pupils commit refined definitions which have no real meaning to them is dwelt upon. In trying to aid the teacher both in methods of instruction and arrangement of material, this book assumes two objects in teaching — to give, first, an insight into the peculiar character of mathematics as a unique domain of human knowledge; second, an understanding of the important rôle it plays because of its numerous practical applications.

The purpose of the second half of the book is to orient the reader as to the essentials, both as to material and as to its arrangement and treatment. No *ideal* course is prescribed, as a course must have personal coloring to be stimulating. But great care is taken everywhere to show the true importance of individual theorems, and give the reader the proper perspective to enable him to choose wisely. Beginning with the triangle, some of the discussion may seem trivial, because we have prejudiced minds as to the proper order and methods. But the succeeding chapter on the theory of parallels is extremely clear, and reveals the thoroughness of the writers. Various methods of approaching the study of the circle are criticised, also that of the equality of plane figures. The teacher must finally answer the double question: What shall I

teach and in what order? A very useful but rather out of place chapter on the theory of irrational numbers is interjected at this point. The theory and application of the Dedekind cut as well as Cantor's theory of irrational numbers are outlined quite fully. Strenuous objection is made to the custom of American texts of applying directly the laws of arithmetical proportions to geometrical quantities. Rational advice is also given against the tendency to number theorems and operations one, two, three, etc., and to compel students to commit this enumeration.

The last seven chapters treat topics rarely found in an elementary American text. The discussion of similarity of plane figures, with some suggested applications, is valuable. The extension of the idea of similarity to circles is very fruitful. Various methods of "measuring" circles are treated, including Huygens' method, although this is pointed out as being of doubtful value in the lower schools. Anharmonic ratios of points on a line and of lines through a point are treated with elegance, and considerable space is devoted to pole and polar with respect to a circle. It seems unfortunate that American texts ignore so largely these fruitful and interesting fields. Pencils and nets of circles, with their properties, are discussed in one chapter and inversion in the plane in another. The last chapter is a long discussion of the Apollonius *tao*-problem. The methods of Apollonius are first discussed, then the modern solutions of Gergonne, Massfeller, and C. Adams are criticised and tested by Study's criteria. The authors' discussion of Study's criteria for the best solution of a construction problem is not without interest. This last half of the book is replete with brief scientific discussions and helpful remarks which the reviewer must refrain from pointing out in more detail. The authors are not entirely free from some of the faults they inveigh against, but the volume doubtless deserves a place among other books of its kind, for it is fearless in its criticisms of others, is thorough in research, and has the merit of bringing the history and bibliography up to date.

D. D. LEIB.

Complementi di Analisi algebrica elementare. By FEDERICO AMODEO. Luigi Pierro, Naples, 1909. 284 pp.

THIS little volume is one of a series of elementary text books by the same author and is intended for use in the Istituto Tecnico of Italy. The author says in the preface that a stu-

dent should know the things contained in this book before he enters the university. This makes the volume interesting, as it shows us what is considered a good university preparation in Italy. This is not so different from our own ideas, but many of the subjects here treated we reserve for a course in college algebra. However, many conceptions which we teach in college algebra or at a later period the Italian student gets before he begins his second course in algebra for which the present book is intended. Some of these conceptions are interval, independent variable, inverse function, geometric representation of a function, and sequence of numbers.

The book contains seven chapters the titles of which will sufficiently indicate the contents: Calculus of combinations; Continued fractions; Analysis of indeterminates of the first degree; Inequalities and systems of inequalities; Discussion of equations and problems and equations of second degree; Finite and continuous functions, limits, indeterminate forms; Maxima and minima, discussion of functions. The book also contains an appendix on geometrical conics. Each chapter closes with an excellent set of problems.

Ordinarily the Italian student devotes much more time to the study of elementary mathematics than the American student. But as here indicated, the time is not spent in anticipating college work, as is often done by preparatory schools in this country, but is devoted to doing more thoroughly the ordinary elementary work.

The presentation shows the author to be a master. But the general appearance of the book from the publisher's point of view would not be considered good in this country.

C. L. E. MOORE.

College Algebra. By H. L. RIETZ and A. R. CRATHORNE.
New York, Herry Holt, 1909. xiv + 261 pp.

THE introduction to this text is concerned with the reasoning in the transition from numerical to literal quantities. Addition and multiplication are regarded as fundamental operations and no attempt is made to define them. Their laws, including their commutative, associative, and distributive properties, are given as assumptions. Subtraction is then defined with reference to addition, and division with reference to multiplication. The algebraic use of these four operations including the laws of real indices is discussed in some detail in the same chapter. The

student's attention is also called to the fact that proofs are quite different from definitions and assumptions, a distinction which he should understand and appreciate at this stage of his mathematical study.

The second chapter provides for a brief review of the student's previous knowledge of algebra. It contains a number of exercises on algebraic reduction, including parentheses, complex fractions, factoring, and radicals. The application of these principles is illustrated in a list of examples calling for the evaluation of a number of builders' and engineers' formulas. The practical nature of these and many subsequent examples is one of the noticeable and commendable features of the book. Chapters three and four take up functional notation, graphs, and equivalent equations. Following this, determinants of the second and third orders are introduced for the purpose of solving systems of linear equations. A more detailed discussion of the elementary principles of determinants is given near the close of the book.

Chapters six to twelve inclusive deal with quadratic equations, maximum and minimum values of quadratic functions of the form $ax^2 + bx + c$, inequalities, mathematical induction, variation, progressions, and complex numbers. The theory of equations is next taken up in a manner worthy of the student's careful attention. One criticism, however, might be offered concerning the article on Horner's method. The graph of the function is emphasized in a way that leads the reader to think of it more as a means of obtaining results than as an interpretation of them. The latter part of the book discusses logarithms, infinitesimals, limits, series, undetermined coefficients, permutations, combinations, probabilities, and determinants.

While recognizing the book as an excellent one for college work, the reviewer offers a few criticisms. In example 13, page 84, a_n must be positive. The sentence in line 7 from the bottom of page 168 might better be made to read "any *finite* number of variables. The use of the word limit in case (c), page 171, is scarcely justified by the author's definition of a limit. The phrase, "If from some point on" in article 135, page 178, is not clear to a person unacquainted with the intended meaning. The expression "— 102" in line 4, page 180, might better be omitted; the sentence is complete and expresses the desired meaning more accurately without it. The theorem in article 141, page 188, should read "if each term is

numerically less." The answer given to example 17, page 213, is incorrect. There are a few typographical errors such as $(2n + 1)$ instead of $(2n - 1)$ in example 2, page 87, but they are so trivial as to give no serious trouble and the book as a whole is quite free from them.

J. V. McKELVEY.

Das Prinzip der Erhaltung der Energie. Von MAX PLANCK.
Zweite Auflage. Leipzig, B. G. Teubner, 1908. xvi + 278 pp.

AMONG the many series of books published under some collective title, one which is at the same time scientific and to a certain extent suited to intelligent popular consumption is *Wissenschaft und Hypothese*. It is the German translation of Poincaré's *La Science et l'Hypothèse*, which has the position of volume I and which apparently has given its title to the whole series. Other volumes which have already appeared are Poincaré's *La Valeur de la Science* (translated), Lipps's *Mythenbildung und Erkenntnis*, Bonola's *Die nichteuklidische Geometrie* (from the Italian), G. Darwin's *Ebbe und Flut* (from the English), Hilbert's *Grundlagen der Geometrie*, Picard's *La Science moderne* (translated), and Planck's *Das Prinzip der Erhaltung der Energie*. From these titles it will appear that the series does not consist chiefly of books written especially for it, but is made up at least in large part of works already written in one language or another and assembled (after translation if need be) into one group. Indeed, some of the volumes were already several years old when the series was started, and among the oldest is Planck's which now celebrates the attainment of its majority with a second edition.

Of such well known and acknowledged value and interest is Planck's discussion of the historical development of the principle of the conservation of energy, of the formation and proof of the principle, and of the different kinds of energy, that little need be said of this second edition, which is in no essential way different from the first. The development of science in the last quarter century has offered much that might be added to the book, but has disclosed nothing that need be taken from it. To this statement one possible exception may be taken in view of Minkowski's recent presentation of electromagnetic and mechanical theories from the point of view of the principle of relativity. In the fundamental matrices which Minkowski sets up, the energy is but one element of many and has no apparent invariance or predominating importance!

E. B. WILSON.

Grundriss der Naturphilosophie. Von W. OSTWALD. Leipzig, Philipp Reclam, jun., 1908. 195 pp.

RECLAM'S Universal-Bibliothek is widely known as a good place to look for almost anything from Bret Harte to Kant with tolerable assurance of finding it attractively put up in small volumes at small prices. Probably somewhere between the rather wide limits just assigned would be placed Ostwald's *Grundriss der Naturphilosophie*; how to place it in much narrower limits is a question; it probably would hardly have appealed to Kant as philosophy or to Bret Harte as nature. There are four parts, general epistemology, logic with the theory of aggregates and mathematics, the physical sciences, and the biological sciences. The point of view throughout is that of the physicist or chemist; the philosophy is not refined, nor the mathematics logical. It is probably precisely these characteristics that will make the book readable and interesting and instructive to the general public. Any admirer of pragmatism will find much here to delight him. To any one not trained in philosophical or mathematical or scientific thinking but desirous of awakening his ideas toward science and philosophy the book is heartily to be recommended as introductory reading.

E. B. WILSON.

NOTES.

The opening (January) number of volume 12 of the *Transactions of the American Mathematical Society* contains the following papers: "An invariantive investigation of irreducible binary modular forms," by L. E. DICKSON; "An application of symbolic methods to the treatment of mean curvatures in hyperspace," by W. H. BATES; "On the order of linear homogeneous groups (fourth paper)," by H. F. BLICHFELDT; "The metrical aspect of the line-sphere transformation," by J. L. COOLIDGE; "Natural systems of trajectories generating families of Lamé," by EDWARD KASNER; "A fundamental system of invariants of the general modular linear group with a solution of the form problem," by L. E. DICKSON; "Linear difference equations and their analytic solutions," by R. D. CARMICHAEL.

Professor C. L. BOUTON has retired from the editorial staff of the *Transactions*. Professors MAX MASON and P. F. SMITH have been added to the staff as associate editors.

THE opening (January) number of volume 33 of the *American Journal of Mathematics* contains the following papers: "On a class of cubic surfaces with curves of the same species," by JOHN EIESLAND; "The automorphic transformations of the binary quartic," by A. H. WILSON; "Theorems on the simple finite polygon and polyhedron," by N. J. LENNES; "On the solutions of certain types of linear differential equations with periodic coefficients," by F. R. MOULTON and W. D. MACMILLAN. The frontispiece is a portrait of Professor ULISSE DINI, of the University of Pisa.

AT the Minneapolis meeting of the American association for the advancement of science it was decided to hold the next meeting of the association at Washington, D. C. Dean C. E. BESSEY, of the University of Nebraska, was elected president of the association. Professor E. B. FROST, director of the Yerkes Observatory was elected vice-president and chairman of Section A. The retiring vice-president, Professor E. W. BROWN, of Yale University, delivered his address on "the relations between Jupiter and the asteroids." The programme of the section consisted of astronomical papers, except at the joint sessions with the Chicago Section of the American Mathematical Society. At the Washington meeting the retiring vice-presidential address will be by Professor E. H. MOORE, of the University of Chicago.

AT the meeting of the London mathematical society held on December 8 the following papers were read: By G. H. HARDY, "Properties of logarithmico-exponential functions," and "Some results concerning the increase of functions defined by an algebraic differential equation of the first degree"; by A. A. ROBB, "Optical geometry of motion"; by T. C. LEWIS, "Note on the Pellian equation," and "A property of the number 7"; by G. B. MATHEWS, "On the arithmetical theory of binary cubic forms."

THE fifteenth meeting of the association of teachers of mathematics of the Middle States and Maryland was held at Philadelphia, November 26. The following papers were read: By

I. J. SCHWATT, "Is the average secondary school pupil able to acquire a thorough knowledge of all the mathematics ordinarily given in those schools?"; by E. H. KOOH, "Training for efficiency in elementary mathematics." Reports by the committee on mathematics in continuation schools and by the committee on a syllabus for algebra were read.

W. H. METZLER was elected president and H. F. HART secretary for the year 1911.

THE ninth regular meeting of the Rochester section of the association of the teachers of mathematics in the Middle States and Maryland was held at Rochester on December 27 in affiliation with the meeting of the New York teachers' association. The following papers were read: By C. W. WATKEYS, "Volumes as functions, with their graphs"; by C. F. WHEELLOCK, "Informal discussion of mathematical work in New York State"; by F. W. OWENS, "On the superposition postulate in geometry"; by W. BETZ, "Causes of failure in mathematics: report prepared for the international commission."

THE Syracuse section of the association of teachers of mathematics in the Middle States and Maryland held its annual meeting at Syracuse in affiliation with various other educational organizations on Friday, December 30, at which the following papers were read: By W. B. CARVER, "Some topics of school mathematics of especial importance to students who expect to study analytic geometry and calculus"; by C. A. SHAVER, "The new syllabus in arithmetic for New York State"; by W. SMITH, "What results are we getting from graphic algebra?"

THE New York section of the Association of teachers of mathematics in the Middle States and Maryland had a general meeting at the Hotel St. Denis in New York city January 6, 1911. After a dinner served to about 125 guests, the members heard the reports on the progress of the International commission on the teaching of mathematics made by the American commissioners, Professor D. E. SMITH, Professor W. F. OSGOOD and Professor J. W. A. YOUNG.

UNIVERSITY OF ILLINOIS. Courses offered in the second semester, 1910-1911. — By Professor S. W. SHATTUCK: Calculus of variations, three hours. — By Professor E. J. TOWNSEND: Functions of a complex variable, three hours. — By

Professor G. A. MILLER : Elementary theory of groups, three hours. — By Professor H. L. RIETZ : Theory of averages and the mathematical theory of investments, three hours. — By Professor C. H. SISAM : Algebraic surfaces, three hours. — By Professor J. B. SHAW : Vector differentials and integrals, three hours. — By Professor A. EMCH : Elliptic functions, three hours. — By DR. A. R. CRATHORNE : Theory of linear differential equations, three hours. — By Dr. G. E. WAHLIN : Partial differential equations, three hours. — By Dr. R. L. BÖRGER : Projective geometry and linear transformations, three hours.

At the annual public meeting of the Paris academy of sciences on December 19 the following announcements were made. The grand prize of the mathematical sciences was not awarded, as no meritorious memoir on the stated problem had been received. The Francoeur prize was awarded to E. LEMOINE; the Poncelet prize to M. RIQUIER; the Binoux prize to E. LEBON. The Montyon prize in mechanics was awarded to J. GAULTIER for his inventions in connection with surveying instruments; the Fourneyron prize was not awarded. The Pierre Guzman prize was not awarded, but the accrued interest was granted to the late M. LÉVY; the Lalande prize was awarded to P. H. COWELL and A. CROMMELIN for their researches in connection with Halley's comet; the Valz prize to S. JAVILLE for his work on nebulae and periodic stars; the Janssen medal to W. W. CAMPBELL for his researches in stellar spectroscopy. The Jerome Ponti prize was awarded to H. ANDOYER for his new trigonometric tables in which the logarithms are carried to 14–17 decimal places.

THE annual meeting of the Italian association for the advancement of science was held at Naples, December 15–21, under the presidency of Professor CIAMICIAN, whose address was on "The cooperation of the sciences."

In pure and applied mathematics the following papers were read: By E. BOMPIANI, "Contribution to projective differential geometry of hyperspaces"; by M. O. CORBINO, "Fifty years after the discovery of Pacinotti's ring"; by A. GARBASSO, "The emission of light"; by A. DE NORA, "Remarks on the Müller-Breslau method of the calculation of frameworks"; by L. SILBERSTEIN, "Mutual mass of two electrons"; by C. SOMIGLIANA, "On elastic properties of the

earth"; by A. TUMMARELLO, "Types of homaloidal surfaces"; by G. VACCA, "History of mathematics in the Far East, contributions by T. HAYASHI and V. MIKAMI."

DR. M. CIPOLLA, of the University of Palermo, has been appointed associate professor of algebra at the University of Catania.

DR. E. VESSIOT, of the University of Lyon, will give a course in mathematical analysis at the University of Paris during the second semester of the present year.

PROFESSOR C. NEUMANN, of the University of Leipzig, will retire from active teaching at the end of the present semester. He will complete his seventy-ninth year on May 7.

MR. H. J. PRIESTLEY, assistant lecturer in mathematics at the University of Manchester, has been appointed professor of mathematics and physics at the new University of Queensland.

PROFESSOR G. CASTELNUOVO, of the University of Rome, and Professor U. DINI, of the University of Pisa, have been elected to membership in the royal institute of Venice.

PROFESSOR M. ABRAHAM, of the technical school of Milan, has been elected a member of the royal institute of Lombardy.

DR. H. LEBESGUE, of the University of Poitiers, has been appointed master of mathematical conferences at the University of Paris.

PROFESSOR L. LECORNU, of the Ecole polytechnique, has been elected a member of the academy of sciences of Paris.

PROFESSOR A. EINSTEIN, of the technical school of Zürich, has been appointed professor of mathematical physics at the German University of Prague.

DR. L. BIEBERBACH has been appointed docent in mathematics at the University of Königsberg.

DR. R. NEUENDORFF has been appointed docent in mathematics at the University of Kiel.

DR. H. TETZE, of the University of Vienna, has been appointed associate professor of mathematics at the German technical school at Brünn.

PROFESSOR G. B. HALSTED has been elected corresponding member of the academy of sciences of Bordeaux.

PROFESSOR JACQUES HADAMARD, of Paris, will deliver at Columbia University in the month of October, 1911, two courses of lectures on mathematics and mathematical physics. A course of weekly lectures open to the mathematical public is also planned. Further details will be announced later.

At the University of California, associate professor G. C. EDWARDS has been promoted to a full professorship of mathematics.

At Princeton University Dr. G. D. BIRKHOFF has been advanced to a full professorship of mathematics.

MR. J. B. SMITH, of the University of Virginia, has been appointed assistant professor of mathematics at Hampden Sydney College.

At the University of Pennsylvania, Dr. H. B. SMITH has been appointed instructor in mathematics to fill the vacancy caused by the temporary absence of Professor H. B. EVANS.

PROFESSOR JULES TANNERY, of the Ecole normale, died November 11, at the age of 72 years.

PROFESSOR K. LASSWITZ, of Gotha, died October 17, at the age of 62 years.

PROFESSOR JULIUS PETERSEN, of the University of Copenhagen, died August 5, 1910, at the age of 71 years.

PROFESSOR TH. N. THIELE, former director of the observatory at Kopenhagen, died September 26, at the age of 71 years.

CATALOGUES of second-hand mathematical books: A. Hermann et Fils, 6 rue de la Sorbonne, Paris, catalogues in exact and natural sciences and in physics, 1910, about 160 entries in mathematics.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

AHRENS (W.). Mathematische Unterhaltungen und Spiele. 2te, vermehrte und verbesserte Auflage. In 2 Bänden. 1ster Band. Leipzig, Teubner, 1910. 8vo. 9+400 pp. M. 7.50.

BERNDT (F.). Grundzüge der Differential- und Integralrechnung. 4te Auflage. Leipzig, Weber, 1910. M. 3.00.

BIEDENKAPP (G.). Sophie Germain, ein weiblicher Denker. Jena, Tauscher, 1910.

BILLY (JACOBI DE). Doctrinae analyticae Inventum novum, Fermats Briefen an Billy entnommen. Herausgegeben und übersetzt von P. von Schaewen. Berlin, Salle, 1910. 143 pp. M. 3.00.

CALCULUS made easy. By F. R. S. London, Macmillan, 1910. 12mo. 188 pp. 2s.

CHARLIER (L.) und ZINNER (E.). Entwicklung des Potentials im Rotationsproblem. Stockholm, Almqvist, 1910.

CLEBSCH (A.). Vorlesungen über Geometrie. Bearbeitet und herausgegeben von F. Lindemann. 2te, vermehrte Auflage. 1ter Band. 1ter Teil. 2te Lieferung. Leipzig, Teubner, 1910. 8vo. pp. 481-768. M. 9.00.

ENRIQUES (F.). Probleme der Wissenschaft. 2 Teile. 1ter Teil: Wirklichkeit und Logik. Übersetzt von J. Grelling. (Wissenschaft) und Hypothese. Band XI, 1.) Leipzig, Teubner, 1910. 10+258 pp. M. 4.00.

ENZYKLOPÄDIE der mathematischen Wissenschaften. 3ter Band: Geometrie. Redigiert von W. F. Meyer. 1ter Teil, 4tes Heft. Leipzig, Teubner, 1910. 8vo. pp. 481-770 M. 9.00.

FORTIN (C.). Théorie et applications élémentaires des vecteurs. Paris, Gigord, 1910. 16mo. 7+65 pp.

HENSEL (K.). Ernst Eduard Kummer und der grosse Fermatsche Satz. (Marburger akademische Reden, Nr. 23.) Marburg, Elwert, 1910. 8vo. 22 pp. M. 0.50.

KLEIN (F.) und SOMMERFELD (A.). Ueber die Theorie des Kreisels. 4tes Heft: Die technische Anwendungen der Kreiseltheorie. Für den Druck bearbeitet und ergänzt von F. Noether. Leipzig, Teubner, 1910. 8vo. 761-966 pp. Cloth. M. 9.00.

KOWALEWSKI (G.). Einführung in die analytische Geometrie. Leipzig, Veit, 1910. 8vo. 7+360 pp. Cloth. M. 11.00.

LINDEMANN (F.). See CLEBSCH (A.).

LORENZ (H.). Einführung in die Elemente der höheren Mathematik und Mechanik. München, Oldenbourg, 1910. 8vo. 5+176 pp. Cloth. M. 2.40.

LÜDERS (O.). Ueber orthogonale Invarianten der bizirkularen Kurven vierter Ordnung. (Diss.) Halle, 1910.

- NEERNST (W.) und SCHOENFLIES (A.). Kurzgefasstes Lehrbuch der Differential- und Integralrechnung mit besonderer Berücksichtigung der Chemie. 6te, vermehrte und verbesserte Auflage. München, Oldenbourg, 1910. 8vo. 12+442 pp. Cloth. M. 14.00.
- D'OCCAGNE (M.). Notions élémentaires sur la probabilité des erreurs. Paris, Gauthier-Villars, 1910. 28 pp.
- PARFENTIEFF (N. N.). Études sur la théorie de la croissance des fonctions (en russe). Kasan, University, 1910.
- PRASAD (G.). A textbook of integral calculus with numerous worked examples. London, Longmans, 1910. 8vo. 5s.
- SCHAEFFEN (P. von). See BILLY (JACOBI DE).
- SCHOENFLIES (A.). See NEERNST (W.).
- SOMMERFELD (A.). See KLEIN (F.).
- WOLFF (G.). Über Kollineationen in der Ebene. (Diss.) Giessen, 1910.
- WOLFF (H.). Formelsammlung zur analytischen Geometrie. Berlin, Maass und Plank, 1910. 8vo. 28 pp. M. 0. 70.
- ZINNER (E.). See CHARLIER (L.).

II. ELEMENTARY MATHEMATICS.

- AMIOT (A.). Trattato di geometria elementare. Nuova edizione, con aggiunte e modificazioni per cura di A. Socci. Settima impressione. Firenze, Le Monnier, 1910. 8vo. 8+402 pp. L. 2.50.
- BECKER (E.). Logarithmisch-trigonometrisches Handbuch auf 5 Decimalen. 3te Ausgabe. Leipzig, Tauchnitz, 1910. 8vo. 16+104 pp. M. 1.20.
- BERICHTE über den mathematischen Unterricht in Österreich. 2tes Heft: von T. Konradt, M. Dolinski, und M. Adamička. Wien, Hölder, 1910. 8vo. 52 pp. M. 1.20.
- BERTRAND (G.). Trattato di algebra elementare. Nuova edizione, con aggiunte e modificazioni per cura di A. Socci. Quattordicesima impressione. Firenze, Le Monnier, 1911. 16mo. 7+531 pp. L. 3.00.
- BOREL (E.) et MONTEL (P.). Algèbre. (Cours de mathématiques Émile Borel.) Paris, Colin, 1910. 16mo. 337 pp. Fr. 3.00.
- BOURLET (C.). Éléments d'algèbre, contenant 561 exercices et problèmes. 7e édition, revue. Paris, Hachette, 1910. 16mo. 296 pp. Fr. 2.00.
- Éléments d'algèbre. Corrigés des 561 exercices avec la collaboration de J. Hulot. 3e édition, revue. Paris, Hachette, 1910. 16mo. 355 pp. Fr. 3.00.
- Éléments de géométrie, contenant 762 exercices. 2e édition, revue. Paris, Hachette, 1910. 16mo. 6+383 pp. Fr. 2.50.
- BRUCE (W. H.) and CONY (C. C.). Elements of plane geometry. Dallas, Tex., Southern Publishing Co., 1910. 12mo. 283 pp. \$0.75.
- CATANIA (S.). Corso di algebra elementare. Volume II. Seconda edizione, corretta e accresciuta. Catania, Giannotta, 1910. 16mo. 278 pp. L. 2.00.
- Trattato di aritmetica ed algebra. Terza edizione, migliorata e corretta. Catania, Giannotta, 1910. 16mo. 16+576 pp. L. 4.50.

- CAYAZZONI (L.) e CERCIGNANI (E.). Libro di geometria del piano. Milano, Albright, 1911. 8vo. 300 pp. L. 2.80.
- CERCIGNANI (E.). See CAYAZZONI (L.).
- CODY (C. C.). See BRUCE (W. H.).
- COURS d'algèbre élémentaire conforme aux derniers programmes de l'enseignement secondaire. Par F. G. M. 6e édition. Paris, Poussielgue, 1911. 8vo. 8+568 pp.
- CRANTZ (P.). Lehrbuch der Mathematik für höhere Mädchenschulen and Lyseen. 1ter Teil. 3te Auflage. Leipzig, Teubner, 1910. 8vo. 6+177 pp. Cloth. M. 2.40.
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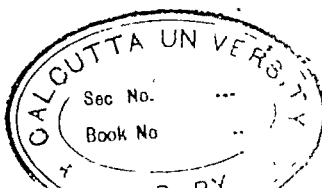
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THE SEVENTEENTH ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE seventeenth annual meeting of the Society was held in New York City on Wednesday and Thursday, December 28-29, 1910, extending through the usual morning and afternoon sessions on each day. The attendance included the following sixty-three members of the Society:

Mr. F. W. Beal, Professor W. J. Berry, Professor G. D. Birkhoff, Professor Maxime Bôcher, Professor Joseph Bowden, Dr. Emily M. Coddington, Professor A. B. Coble, Professor F. N. Cole, Dr. J. R. Conner, Dr. G. M. Conwell, Professor J. L. Coolidge, Professor E. W. Davis, Professor F. F. Decker, Dr. L. S. Dederick, Professor W. P. Durfee, Professor L. P. Eisenhart, Professor J. C. Fields, Professor H. B. Fine, Professor W. B. Fite, Professor A. S. Gale, Professor O. E. Glenn, Professor C. C. Grove, Professor C. O. Gunther, Professor E. L. Hancock, Professor J. G. Hardy, Professor C. N. Haskins, Professor H. E. Hawkes, Dr. W. A. Hurwitz, Professor L. A. Howland, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Dr. D. D. Leib, Dr. N. J. Lennes, Mr. Joseph Lipke, Professor W. R. Longley, Mr. A. R. Maxson, Professor Helen A. Merrill, Dr. H. H. Mitchell, Professor C. L. E. Moore, Professor C. N. Moore, Professor Frank Morley, Professor Richard Morris, Professor G. D. Olds, Dr. H. B. Phillips, Dr. H. W. Reddick, Professor R. G. D. Richardson, Miss S. F. Richardson, Mr. L. P. Siceloff, Mr. C. G. Simpson, Dr. Clara E. Smith, Mr. F. H. Smith, Professor P. F. Smith, Mr. W. M. Smith, Dr. W. M. Strong, Dr. M. O. Tripp, Professor C. B. Upton, Professor Oswald Veblen, Mr. H. E. Webb, Professor H. S. White, Mr. E. E. Whitford, Professor E. B. Wilson, Professor T. W. D. Worthen.

President Maxime Bôcher took the chair at the opening session, yielding it later to Professor Kasner, Ex-President Osgood, and at the morning session on Thursday to the President elect, Professor H. B. Fine. The Council announced the election of the following persons to membership in the Society: Professor Percy Hodge, Columbia University; Mr. C. G. P. Kuschke, University of California; Professor Marion B. White, University of Kansas. Eight applications for membership were received.



On Wednesday evening thirty of the members gathered at the annual dinner, which has for many years been a pleasant adjunct to the regular sessions.

It was decided to hold the summer meeting in 1911 at Vassar College. Professors H. S. White, P. F. Smith, and the Secretary were appointed a committee to make the necessary arrangements. The summer meeting of 1912 will be held at the University of Pennsylvania.

The report of the Treasurer, Auditing Committee, and Librarian are published in the Annual Register. The membership of the Society has increased during the past year from 618 to 641, including at present 60 life members. The total attendance of members at all meetings was 317. The number of papers presented was 145. In the annual election 202 votes were cast. The Society's library shows the normal yearly increase, the total number of books on the shelves being now 3,508. The Treasurer's report shows a balance of \$8,124.53, a slight increase over last year, although the subventions received from various universities for the support of the *Transactions* have been discontinued, and the Society has paid a first installment of 1,000 francs of its contribution toward the publication of the collected works of Euler. The income from sales of the Society's publications during the year was \$1,660.21. The life membership fund now amounts to \$3,901.83.

During the past sixteen years the Society has expended for printing the BULLETIN, *Transactions*, and other publications \$42,473.57. The total returns from publications have been in the same period \$15,375.20. The university subventions have amounted to \$8 100. Editorial and administrative expenses have been \$11,779.49. The stock of publications on hand may be fairly valued at about \$10,000.

At the annual election, which closed on Thursday morning, the following officers and other members of the Council were chosen :

<i>President,</i>	Professor H. B. FINE.
<i>Vice-Presidents,</i>	Professor G. A. BLISS, Professor W. E. STORY.
<i>Secretary,</i>	Professor F. N. COLE.
<i>Treasurer,</i>	Professor J. H. TANNER.
<i>Librarian,</i>	Professor D. E. SMITH.

Committee of Publication,
 Professor F. N. COLE,
 Professor E. W. BROWN,
 Professor VIRGIL SNYDER.

Members of the Council to serve until December, 1913,
 Professor H. F. Blichfeldt, Professor C. J. Keyser,
 Professor J. L. Coolidge, Professor J. W. Young.

The following papers were read at this meeting:

(1) Mr. S. CHAPMAN: "A note on the theory of summable integrals."

(2) Dr. H. H. MITCHELL: "Concerning a rotation group in six-space."

(3) Professor VIRGIL SNYDER: "An application of a (1-2) quaternary correspondence."

(4) Rev. A. S. HAWKESWORTH: "Three new dimension theorems."

(5) Professor W. R. LONGLEY: "Singular points on the discriminant locus of an ordinary differential equation."

(6) Professor R. G. D. RICHARDSON: "Theorems of oscillation for two self-adjoint linear differential equations of the second order with two parameters."

(7) President MAXIME BÔCHER: "A simple proof of a fundamental theorem in the theory of integral equations."

(8) Professor J. L. COOLIDGE: "The metrical aspect of the line-sphere transformation."

(9) Dr. E. J. MILES: "The absolute minimum of a definite integral in a special field."

(10) Professor J. C. FIELDS: "A method of proving certain theorems relating to rational functions which are adjoint to an algebraic equation for a given value of the independent variable."

(11) Professor F. F. DECKER: "Concerning the order of a restricted system of equations."

(12) Professor PAUL SAUREL: "On the classification of crystals."

(13) Professor A. B. COBLE: "An application of Moore's cross ratio group to the solution of the sextic equation."

(14) Professor A. B. COBLE: "The cubic surface and plane six-point."

(15) Professor C. L. E. MOORE: "Conjugate directions on a hypersurface in S_4 and some allied curves."

(16) Professor W. H. BATES: "An application of symbolic methods to the treatment of mean curvature in hyperspace."

(17) Dr. H. B. PHILLIPS: "The Galois theory of sets of multipartite variables."

(18) Dr. J. R. CONNER: "Correspondences associated with the rational plane quintic curve."

(19) Professor L. P. EISENHART: "Conjugate systems and envelopes of spheres."

(20) Mr. JOSEPH LIPKE: "Natural families of curves in a general curved space of n dimensions."

(21) Professor O. E. GLENN: "On the discriminants of ternary forms."

Mr. Chapman's paper was presented to the Society and read by Professor C. N. MOORE. In the absence of the authors, the papers of Professor Snyder, Mr. Hawkesworth, Dr. Miles, Professor Saurel, and Professor Bates were read by title. Professor Bates's paper has appeared in full in the January *Transactions*. Professor Saurel's paper will be published in the *BULLETIN*. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. The main result of Mr. Chapman's paper is a proof of the following theorem: If the integral

$$\int_a^\infty f(x) dx$$

is summable (C_r) , i. e., if

$$\lim_{x \rightarrow \infty} \frac{r!}{x^r} \int_a^x \int_a^{\alpha_1} \int_a^{\alpha_2} \dots \int_a^{\alpha_r} f(\theta) d\theta d\alpha_r \dots d\alpha_2 d\alpha_1$$

exists, and if furthermore $f(x)/x^r$ is uniformly continuous in the interval $x \geq k > 0$, then $\lim_{x \rightarrow \infty} f(x)/x^r = 0$.

For the case $r = 1$, this theorem reduces to one due to C. N. Moore,* except that the condition in the latter theorem that $f(x)$ should be uniformly continuous is replaced by the less stringent condition that $f(x)/x$ should have that same property.

2. A collineation group in ordinary three-space, when represented on the Plücker line coordinates, is a linear group in six variables with a quadratic invariant. If a transformation be effected so that the quadratic invariant becomes

$$\sum_{i=1}^6 x_i^2 = 0,$$

* Cf. *Transactions*, vol. 3 (1907), p. 314.

the group becomes one of orthogonal substitutions, and may be regarded as a group of rotations about a point in six-dimensional space.

Groups of order 2520 and 11520 in three-space correspond to groups leaving invariant regular solids in six-space. The group of order 25920 has not this property, but is shown by Dr. Mitchell to leave invariant a semi-regular solid of 27 vertices, bounded by 27 regular five-dimensional solids of 10 vertices and 72 regular five-dimensional solids of 6 vertices.

3. The (1, 2) correspondence discussed by Professor Snyder is defined by a triply infinite system of quadric surfaces passing through six basis points. The surfaces of this system passing through a seventh point P will also pass through a second point Q . A three-dimensional involution I exists between P and Q . If the parameters in the equation of the quadrics are regarded as plane coordinates in a second space, a (1, 2) point correspondence between the two spaces is established. The surface of branch points in the second space is a general Kummer surface, and the locus of coincident points in the first is the Weddle surface. A number of birational transformations are discussed and the conditions that the basis points are in one, two, ..., six fold involution are determined.

4. Mr. Hawkesworth presented the three following dimension theorems:

(1) The number of A -dimensional boundaries of an N -dimensional rectangular figure is

$$\frac{N!}{(N-A)!A!} \cdot 2^{N-A}.$$

(2) The number of A -dimensional boundaries of an N -dimensional tetrahedroidal figure is

$$\frac{(N+1)!}{(N-A)!(A+1)!}.$$

From this it follows that the numerical values of the boundaries must rise and fall symmetrically to a medial value, the number $N+1$ of the vertices equalling that of the last $N-1$ boundaries, the edges and the $N-2$ boundaries being alike $\frac{1}{2}(N+1)N$, and the planes and the $N-3$ boundaries each being $(N+1)N(N-1)/3!$, and so on.

These two formulas hold true, even when A is taken greater than N , the negative and reciprocal result, as for example that there are $\frac{1}{6}$ cubes in a square, meaning simply that, reversely and reciprocally, there are 6 squares in a cube.

(3) In any N dimensional rectilinear figure where N is even the boundaries, taken in their sequence alternately plus and minus (i. e., + vertices — edges + planes — solids + ...), always sum up to zero. But if N be an odd number, then the summation gives + 2.

Or, if we count the N -dimensional figure as one of its own boundaries, and thus add -1 or $+1$ to the summation, according as N is odd or even, then the summation is always $+1$ for all rectilinear figures. In this case the binomial development of $(2 - 1)^N$ will give us both the sequence and the summation of the boundaries of a rectangular N -dimensional figure, while in a similar way the symmetrical sequence and summation of those of an N -dimensional tetrahedroidal figure can be represented by $-(1 - 1)^{N+1}$, omitting the first term of the binomial development.

5. The discriminant of an ordinary differential equation of the second degree, $Ay'^2 + 2By' + C = 0$, is $\Delta = B^2 - AC$. It is supposed that, within the region considered, the coefficients are analytic functions of x and y , and that $y = \eta(x)$ is an analytic branch Γ of the discriminant locus $\Delta = 0$. If the points at which both A and C vanish are excluded, the nature of the solutions in the neighborhood of any point on Γ is known. The object of Professor Longley's paper is an investigation of the integral curves through a point P on Γ at which A , B , and C vanish. It is found that an infinite number of integral curves may pass through P and the nature of these curves is not determined by the nature of Γ . For example, Γ may be a singular solution. Then through every point except P there passes one other integral curve, and it is tangent to Γ ; while an infinite number of integral curves may pass through P no one of which is tangent to Γ . Or Γ may be a cusp locus, so that through every point except P there passes one integral curve, and it has a cusp at this point; while an infinite number of integral curves may pass through P no one of which has a cusp at P . In this case it may happen also that for any point except P the integral curve is not tangent to Γ , while at P every one of an infinite number is tangent to P .

6. Klein has considered the problem of determining the two parameters of the differential equation of Lamé in such a manner that the equation furnishes two solutions, one of which vanishes at the end points of a certain interval a_1b_1 and oscillates m times within the interval, and the other vanishes at the end points of a second interval a_2b_2 and oscillates n times in the interval, m and n being positive integers or zero arbitrarily prescribed. Other special problems of a special nature which involve two or more parameters have been treated by Klein and Bôcher.* For the case of two parameters these may be considered as special cases of the following: If $p_i(x) > 0$, $q_i(x) \leq 0$, $A_{ij}(x)$, ($i = 1, 2; j = 1, 2$) are given analytic functions of x , when can the parameters λ, μ be so determined that the equations

$$(p_1u')' + q_1u + (\lambda A_{11} + \mu A_{12})u = 0, (p_2v')' + q_2v + (\lambda A_{21} + \mu A_{22})v = 0$$

have solutions $u(x), v(x)$ which satisfy the boundary conditions

$$u(a_1) = u(b_1) = 0, v(a_2) = v(b_2) = 0$$

and oscillate m and n times within the intervals a_1b_1 and a_2b_2 respectively (the intervals a_1b_1 and a_2b_2 may or may not coincide)? By transforming the two equations into a partial differential equation with one parameter, Hilbert showed in a paper read before the Mathematische Gesellschaft zu Göttingen that in case

$$p_1 = p_2 = 1, q_1 = q_2 = 0, A_{12} = A_{21} = 1, A_{11} > 0, A_{22} < 0$$

there exist an infinite number of parameter pairs $\lambda_1, \mu_1; \lambda_2, \mu_2; \dots$ for which solutions exist.

Professor Richardson has attacked the problem from the standpoint of the calculus of variations and shows that in case $A_{12} > 0$, $A_{22} < 0$, and $A_{11}(x)A_{22}(x) - A_{12}(x)A_{21}(x)$ is not identically zero there exist parameter values λ, μ such that the solutions $u(x), v(x)$ oscillate m and n times respectively. He shows further that the oscillation theorem holds true when the intervals coincide and A_{12} and A_{22} have the same sign throughout the interval while $A_{11}A_{22} - A_{21}A_{12}$ is different from zero. The results hitherto obtained for problems of this type are special cases of these theorems.

7. The theorem considered in Professor Bôcher's paper states that a homogeneous linear integral equation of the second kind can have only a finite number of linearly independent

*Cf. article by Bôcher, *Encyklopädie der mathematischen Wissenschaften*, II A 7a.

solutions, and gives an explicit formula bounding this number. The proof depends only on the well known fact that the Gramian of a set of functions is positive or zero according as these functions are linearly independent or dependent.

8. The line-sphere correspondence of Lie owes its importance largely to the fact that it is a contact transformation, intersecting lines corresponding to tangent spheres. In Professor Coolidge's paper it is shown that this is merely a special case of a more general relation connecting distances of lines and angles of spheres. Applications are made to the linear complex and congruence, and the dual projective geometry of two ternary domains. The paper has been published in the January *Transactions*.

9. In Dr. Miles's paper, the absolute minimum of a definite integral

$$J = \int_{t_0}^{t_1} F(x, y, x', y') dt$$

in a special field is discussed. Under the assumptions ordinarily imposed on the function F and the admissible curves it is known that there exists a one-parameter family of extremals passing through the point 0. This family has in general an envelope and it is here supposed that the envelope has a cusp. The covering of the region near the cusp point is considered. It is shown that outside the V-shaped cusp region there is a single extremal through each point, while for points within there are three, one of which has already touched the envelope. Then the field about an extremal arc whose end points are conjugate is discussed. The assumption is made that the conjugate point is a cusp point in the envelope, and it is found that the field is a three-sheeted affair similar to a Riemann surface, the sheets being joined by the two branches of the envelope. Further the envelope law holds along either branch of the envelope. A curve M in the V-shaped cusp region is then determined which marks the cessation of absolute minimum. This curve is continuous, passes through the cusp point 1, and is such that the extremal ceases to furnish an absolute minimum as soon as it meets this curve. Finally sufficient conditions for an absolute minimum in this field are considered.

10. In Professor Fields' paper proofs were given of the following theorems:

(1) The degree of a rational function which is adjoint to an algebraic equation for the value $z = \infty$ must be less than the degree of the equation written in the form

$$v^n + F_{n-1}v^{n-1} + \dots + F_0 = 0,$$

and the degree of the element of the function involving the power v^{n-1} must be $\leq n - 1$.

(2) In the case of an integral algebraic equation, an integral rational function must be divisible by the factor $z - a$ if its order of coincidence with the n branches corresponding to the value $z = a$ severally exceed by 1 the orders of coincidence defining adjointness for these branches.

(3) If an integral rational function is adjoint for $n - 1$ of the n branches of an integral algebraic equation corresponding to a value $z = a$, it must also be adjoint for the remaining branch.

The principle employed in the proof of each of the above theorems is the same. It is possible in a certain manner to construct rational functions which for a given value of the variable z possess any desired set of adjoint orders of coincidence. In the case of a finite value $z = a$ these special functions are integral. In any case they conform to the requirements of the theorem in question. Now if there exists a function which fails to accord with one of the theorems, the sum obtained on adding to this function one of the corresponding special functions will also fail to accord with the theorem. Proper selection however of the function to be added would, in the case of either of the first two theorems, give a sum whose orders of coincidence with the n branches are indefinitely great, while in the case of the third theorem the orders of coincidence of the corresponding sum with $n - 1$ of the branches would be indefinitely great. Such orders of coincidence are readily shown in each case to be compatible only with the requirements of the theorem in question. This however is in contradiction to our first deduction that the sum of a function which does not accord and a function which does accord with one of the theorems does not itself accord with the theorem. It follows therefore that no function to which any one of the theorems has reference fails to conform to that theorem.

11. Professor Decker's paper establishes a method for the determination of the order of the restricted system of equations which arise from putting equal to zero the determinants obtained

by suppressing each set of $n-m$ columns of a matrix of m rows and n columns. The method of treatment is based on the linear independence of certain of the determinants and the expression of the others in terms of them. The apparent order of the original system is found to depend upon that of a second and this in turn upon that of a third, etc., until finally a reduction formula is established by means of which the determination of the order is made to depend upon the orders of determinants.

The order is calculated for certain cases. These results specialized with regard to the degree of the elements compared with Segre's specialized with regard to the order of the determinants are found to agree. They also confirm the statement of Salmon.

13. The object of Professor Coble's first paper is to estimate the utility of E. H. Moore's cross ratio group for the solution of the sextic. The group appears as the natural bond between the given sextic and a resolvent sextic of the general diagonal type. A material algebraic advantage is gained by using this resolvent rather than the given sextic.

14. In Professor Coble's second paper some formulas of the first are employed to obtain an equation of a cubic surface mapped from a plane by means of cubic curves on a given six-point. The equations of the forty-five triple tangent planes of the surface are derived. An interesting further result is a complete system of invariants for the plane six-point.

15. Two consecutive tangent S_3 's to a hypersurface in S_4 will intersect in a plane which is tangent to the hypersurface. This plane and the line joining the points of contact are said to be conjugate. In this paper Professor Moore has discussed this correspondence, also the correspondence between the line of intersection of three consecutive tangent S_3 's and the plane determined by the points of contact.

17. In the usual treatment of the Galois theory of equations the property of $x_1, \dots, x_n$ of being roots of an algebraic equation in one variable is not essential. These quantities may therefore be generalized without affecting the validity of many results in that theory. In the paper of Dr. Phillips they are considered as multipartite numbers representing points in a plane or space of higher dimensions.

18. Making use of the properties of the rational norm-curve in a space of five dimensions, Dr. Conner proves the existence of certain multiple correspondences between the planes of the two curves $\rho\alpha_i = (\alpha_i t)^5$; $\rho\eta_i = (b_i t)^5$ ($i = 0, 1, 2$), where the b 's are three linearly independent forms apolar to the α 's. The significance of these correspondences for the general rational plane quintic curve is then discussed.

19. Darboux has considered two parameter families of spheres which possess the property that the lines of curvature on the two sheets of the envelope correspond, and he has shown that the surface of centers of the spheres is cut in a conjugate system by the developables of the congruences of normals to the two sheets. Guichard has called a conjugate system of this sort a "reseau 20." Professor Eisenhart considers the determination of these systems on any surface. It is reducible to the solution of a system of linear partial differential equations of the first order. All conjugate systems with the same Gaussian representation as a "reseau 20" are likewise of this type, and with one of them there is associated a cyclic system whose circles pass through a point. This unique surface may be taken as characteristic of the group. Several interesting particular cases are discussed, one of them leading to results concerning surfaces of Monge.

20. In a paper read at the Princeton summer meeting of the Society, Mr. Lipke gave the complete geometric characterization of natural families of curves on a general surface in euclidean space of three dimensions. Natural families of curves are defined as the systems of extremals connected with variation problems of the type $\int F ds = \text{minimum}$, where ds is the element of arc length in the space considered, and F is any point function. It is the purpose of the present paper to give the complete geometric characterization of natural families of curves on a general hypersurface V_n in a euclidean S_{n+1} . The characteristic properties are: (1) the locus of the centers of geodesic curvature of the ∞^{n-1} curves passing through a point is a euclidean S_n ; (2) the n directions in which the osculating geodesic circles (circles of constant geodesic curvature) hyperosculate are mutually orthogonal. The natural families are a special type of a much larger class of curves which are characterized by property (1). We find that our two-dimensional results are included in our n -dimensional results.

21. The ordinary discriminant of the ternary quadratic form $a_x^2 + b_x x_3 + c x_3^2$ is expressible as the resultant of the binary forms a_x^2 , b_x plus c times the discriminant of a_x^2 . Professor Glenn proves the corresponding general theorem for the ternary form of order m , and derives explicit formulas for the $\frac{1}{2}m(m-1)$ ordinary discriminants (conditions that the forms degenerate into distinct linear factors) in terms of the coefficients of the form. One of these discriminants is a linear expression in the resultant of a_x^m and b_x^{m-1} and the discriminant of a_x^m , while the rest are obtained from this one by operating by powers of

$$\Delta_1 = m a_0 \frac{\partial}{\partial b_0} + (m-1) a_1 \frac{\partial}{\partial b_1} + \cdots + a_{m-1} \frac{\partial}{\partial b_{m-1}}$$

and

$$\Delta_2 = m a_m \frac{\partial}{\partial b_{m-1}} + (m-1) a_{m-1} \frac{\partial}{\partial b_{m-2}} + \cdots + a_1 \frac{\partial}{\partial b_0}.$$

He also considers the explicit form of the satellite form of the m -ic and its character as an invariant.

F. N. COLE,
Secretary.

THE WINTER MEETING OF THE CHICAGO SECTION.

THE twenty-eighth regular meeting of the Chicago Section of the American Mathematical Society was held at the University of Minnesota, Minneapolis, Minn., on Wednesday, Thursday, and Friday, December 28-30, 1910, in affiliation with the American Association for the Advancement of Science. Five half-day sessions were held, beginning Wednesday afternoon with a meeting in connection with Section A (mathematics and astronomy) to hear the address of the retiring vice-president, Professor E. W. Brown of Yale University, on "The relations of Jupiter with the asteroids." Other papers of a mathematical character at this session and at its continuation on Thursday morning were by Professor F. R. Moulton of the University by Chicago on "The debt of mathematics to astronomy," and of Dr. W. D. Macmillan of the University of Chicago on "An integrable case in the problem of three bodies." The vice-president of Section A, Professor E. H. Moore of the University of Chicago, presided throughout this session.

The last half-day session, on Friday afternoon, was also a joint meeting with Section A and with Sections B (physics) and D (mechanical engineering) of the American Association, to hear the report of Professor E. V. Huntington of Harvard University, chairman of the committee of twenty appointed at a similar meeting three years ago at Chicago, on "The teaching of mathematics to students of engineering." The work of this committee has extended along two distinct lines, first, the gathering of statistics on this subject from about forty schools where engineering is taught in this country, and secondly, the compilation of a set of syllabi on algebra, geometry, trigonometry, dynamics, and numerical computation. The statistics were turned over to a similar committee working under the International Commission on the teaching of mathematics, and the syllabi formed the basis of Professor Huntington's report at this meeting. They have been printed in preliminary form through the cooperation of the Society for the Promotion of Engineering Education and were presented to all members of this conference for criticisms and suggestions either at the meeting or at any time later to the chairman; for this purpose additional copies may be had on request from the chairman. About forty engineers, physicists and mathematicians attended this meeting and a most interesting and animated discussion was participated in by a large number including Professors E. J. Wilczynski, University of Chicago; A. G. Hall, University of Michigan; E. H. Comstock, School of Mines, University of Minnesota; G. A. Miller, University of Illinois; A. E. Haynes, College of Engineering, University of Minnesota; E. F. Nichols, president of Dartmouth College; T. F. Holgate, Northwestern University; Alexander Ziwet, University of Michigan; E. R. Maurer, College of Engineering, University of Wisconsin; Henry Crew, Northwestern University; H. E. Slaughter, University of Chicago; B. L. Newkirk, University of Minnesota; E. F. Chandler, University of North Dakota; J. J. Flather, head of the department of mechanical engineering, University of Minnesota, and L. E. Dickson, University of Chicago, who presided at the meeting. Also numerous criticisms of the report were presented in writing either directly to the chairman or through the secretary of the meeting. Among these were communications from President R. S. Woodward of Washington, and Professor William Kent of New York. In conclusion it was resolved that the members

of this body tender hearty thanks to Professor Huntington for the great interest which he has shown in this work and the untiring service which he has given to it, that they commend the work of the whole committee for the preparation of a report which it is believed must operate for betterment along the lines of the recommendations, and while not prepared to approve in all respects the details, especially in the syllabus on dynamics, as shown by the full and free discussion at this meeting, yet they heartily endorse the spirit of the report and thank the officers of the Society for the Promotion of Engineering Education who have shown their friendly cooperation in offering to publish these syllabi in their official Bulletin for the purpose of drawing out further criticisms and suggestions either in printed papers or in written communications to the chairman of the committee. The further report of this committee is to be presented at the next meeting of the Society for the Promotion of Engineering Education in June, 1911.

The remaining three sessions of the Chicago Section were devoted to the reading of mathematical papers. Professor L. E. Dickson, chairman of the section, presided except when relieved, during the reading of his own papers, by Professor Alexander Ziwet and by Professor A. G. Hall. The total attendance of forty-five at these sessions included the following thirty members of the society:

Professor G. N. Bauer, Professor W. E. Brooke, Professor E. W. Brown, Professor G. L. Brown, Professor W. H. Bussey, Professor E. H. Comstock, Professor H. H. Dalaker, Professor L. E. Dickson, Professor J. F. Downey, Professor H. T. Eddy, Professor Peter Field, Professor A. G. Hall, Professor G. W. Hartwell, Professor T. F. Holgate, Professor E. V. Huntington, Professor Kurt Laves, President E. O. Lovett, Professor G. A. Miller, Dr. W. D. MacMillan, Professor E. H. Moore, Professor F. R. Moulton, Dr. L. I. Neikirk, Professor B. L. Newkirk, Professor W. J. Risley, Professor R. R. Shumway, Professor H. E. Slaught, Dr. H. L. Slobin, Dr. E. H. Taylor, Professor E. J. Wilczynski, Professor Alexander Ziwet.

On Thursday evening about one hundred of the astronomers, engineers, physicists, and mathematicians dined together at the Commercial Club and fulfilled in every respect the desire, expressed at the April meeting of the Chicago Section, for a closer affiliation between these representatives of the American

Association. Professor E. H. Moore acted as toastmaster and impromptu addresses were made by Dean F. C. Shenehon, of the School of Engineering, University of Minnesota, and President C. S. Howe, of Case School of Applied Science, representing the engineers; by President E. F. Nichols, of Dartmouth College, and Professor E. L. Nichols, of Cornell University, representing the physicists; and by Professor E. V. Huntington, of Harvard University, and Professor E. W. Brown, of Yale University, representing the mathematicians.

The election of officers of the Section for the ensuing year resulted as follows: chairman, Professor L. E. Dickson, secretary, Professor H. E. Slaught, third member of the program committee Professor J. B. Shaw. It was voted that the chairman of the Section be requested to deliver an address on the occasion of retiring from his office at the winter meeting in 1911.

The following papers were read at this meeting:

- (1) Professor G. A. MILLER: "On the use of the co-sets of a group."
- (2) Professor L. E. DICKSON: "Congruential theory of functions of several variables."
- (3) Professor L. E. DICKSON: "Generalization of theorems on linear algebras."
- (4) Professor JACOB WESTLUND: "On primitive roots of ideals."
- (5) Dr. A. B. FRIZELL: "The problem of defining the set of real numbers."
- (6) Professor FLORIAN CAJORI: "A historical note on the Newton-Raphson method of approximations."
- (7) Professor E. J. WILCZYNSKI: "One-parameter families and nets of plane curves."
- (8) Dr. H. T. BURGESS: "Circular numbers for a plane curve."
- (9) Dr. H. T. BURGESS: "Rational anharmonic curves upon a quadric."
- (10) Dr. H. T. BURGESS: "The applications of matrices to cubic forms."
- (11) Professor W. J. RISLEY and Professor W. E. MACDONALD: "Envelopes of one-parameter families of plane curves."
- (12) Dr. W. D. MACMILLAN: "A reduction of two power series in many variables to two equivalent polynomials."
- (13) Dr. H. F. MACNEISH: "The path of light in a medium homogeneous in concentric spherical layers."

(14) Professor KURT LAVES: "The curves of equal action for elliptical coordinates."

(15) Professor L. C. KARPINSKI: "Robert of Chester's translations of the algebra of Al-Khowarazmi."

(16) Professor L. C. KARPINSKI: "Hindu numerals in the Kitab al Fihrist."

(17) Dr. ARTHUR RANUM: "Ruled surfaces and planed hypersurfaces in four dimensional space."

(18) Dr. L. I. NEKIRK: "Transformation groups and substitutions of an infinite degree."

(19) Professor J. B. SHAW: "Use of quaternions in differential geometry."

(20) Dr. H. L. SLOBIN: "On plane quintic curves."

(21) Professor F. R. MOULTON: "On the construction of a certain class of periodic solutions of the problem of three bodies."

(22) Professor F. B. WILLIAMS: "Curves on quintic scrolls."

Professor Williams was introduced by Professor G. A. Miller. In the absence of the authors the papers of Professors Cajori, Westlund, Karpinski, Shaw, and Williams, Drs. Frizell, Burgess, MacNeish, and Ranum were read by title. Abstracts of the papers follow below, the numbers corresponding to those in the above list.

1. If H is any subgroup of the group G , the operators $S_2, S_3, \dots, S_\rho$ may be so selected that every operator of G is found once and only once in each of the following two sets:

$$\begin{aligned} H + HS_2 + HS_3 + \dots + HS_\rho, \\ H + S_2H + S_3H + \dots + S_\rho H. \end{aligned}$$

The sets $HS_\alpha, S_\alpha H$ ($\alpha = 2, 3, \dots, \rho$) are called the co-sets of G as regards H , the former being called right co-sets and the latter left co-sets. When H is added to these co-sets they are called respectively right and left augmented co-sets. The operators of G may also be represented uniquely in each of the following two ways, H_1 and H_2 being two arbitrary subgroups of G :

$$\begin{aligned} H_1H_2 + H_1S_2H_2 + \dots + H_1S_\lambda H_2, \\ H_2H_1 + H_2S_2^{-1}H_1 + \dots + H_2S_\lambda^{-1}H_1. \end{aligned}$$

The sets $H_1S_\alpha H_2$ ($\alpha = 2, 3, \dots, \lambda$) are called double co-sets of G as regards H_1 and H_2 , and when H_1H_2 is added they are called the augmented double co-sets as regards these subgroups.

The objects of Professor Miller's paper are to exhibit how a number of known theorems may readily be proved and related by employing these concepts explicitly, and especially to derive some important relations between abstract groups and substitution groups. The following theorems are proved: (1) If G is represented as a transitive substitution group K , and if K_1 is the subgroup composed of all the substitutions of K which omit a given letter, then any subgroup K' of K has the same number of transitive constituents as the subgroup which corresponds to K_1 has when K is represented with respect to K' . Moreover, the transitive constituents in the two given subgroups have the same relative degrees in the two given representations of G . (2) The product $H_1 H_2$ cannot involve the inverse of each operator in a double co-set with respect to H_1, H_2 .

2. The term function is used, for brevity, in Professor Dickson's first paper to designate a rational integral function with integral coefficients. If p is a prime, two functions f and g of a single variable x have, modulo p , a greatest common divisor $d(x)$ which is unique apart from a constant factor and may be determined by an obvious modification of Euclid's algorithm. We have only to multiply the divisor by such a constant that the new leading coefficient is $\equiv 1 \pmod{p}$. The same method does not apply for functions of two or more variables, nor are the resulting theorems entirely analogous to those holding for a single variable. In the latter case $d(x)$ is a linear combination of f and g modulo p . But for $f = (x+1)y - 1$, $g = xy + x$ and $p = 3$, $xf - (x+1)g = -x(x+2)$, so that the greatest common divisor of f and g is a constant prime to 3; and we do not have $1 \equiv Af + Bg \pmod{3}$, since for $x \equiv 1$, $y \equiv -1$, we have $f \equiv g \equiv 0 \pmod{3}$.

To construct a congruential theory of functions of two variables (and then by induction for n variables), denote by $d(x)$ the greatest common divisor modulo p of the coefficients $f_i(x)$ in $f = \sum f_i(x)y^i$ and call f primitive in y modulo p when $d(x)$ is a constant not divisible by p . We readily show that if fg is divisible modulo p by a function $P(x)$, irreducible modulo p , either f or g is divisible modulo p by $P(x)$. It follows that the product of two functions primitive in y modulo p is primitive in y . If $\phi(x)F$ is divisible modulo p by G , where F and G are primitive in y , then F is divisible by G . A function primitive in y modulo p has as divisors only functions primitive in y .

To establish the existence of a greatest common divisor (f, g) modulo p of f and g , let $\mathcal{J} \equiv N(\text{mod } p)$, $N = Vy^n + V'y^{n-1} + \dots$, where $V, V', \dots$ are functions of x only, and $V \not\equiv 0 (\text{mod } p)$. For a suitable integer k , we have $V^k f = Nq_1 + r_1$, algebraically, where r_1 is of degree $< n$ in y . In this way we get

$$V^k f \equiv gq_1 + r_1, \quad V_1^k g \equiv r_1 q_2 + r_2, \dots (\text{mod } p).$$

Let r_n be the last remainder not divisible by p . Let $f = \phi(x)F$, $g = \gamma(x)G$, $r_n = \rho(x)R$, where F, G, R are primitive in y modulo p . By use of the preceding lemmas, we readily prove that $R = (F, G)$, $(f, g) = G(\phi, \gamma)$. We may now establish the unique factorization theorem.

The same method is applicable to the theory of algebraic forms; in particular, it avoids the use of Kronecker's theorem concerning the coefficients of the product of two polynomials.

3. The second paper by Professor Dickson deals with linear algebras in which multiplication is not assumed to be associative or commutative. Given two numbers X and Y of the algebra, $X \neq 0$, we can determine a number X' such that $XX' = Y$ or $X'X = Y$ if and only if the right hand or left hand determinant

$$\Delta(X) = \left| \sum_{i=1}^n x_i \gamma_{ijk} \right|, \quad \Delta'(X) = \left| \sum_{i=1}^n x_i \gamma_{ik} \right|$$

is not zero. Here the γ_{ijk} are the constants of multiplication. We assume that the unit e_1 is a modulus. Then x_1 occurs only in the terms of the main diagonal of Δ or Δ' . Replacing x_1 by $x_1 - \rho$, we obtain the right and left hand characteristic determinants of X

$$\Delta(X - \rho) = \sum_{i=0}^2 r_i \rho^i, \quad \Delta'(X - \rho) = \sum_{i=0}^n l_i \rho^i.$$

Set $X_1 = X$, $X_2 = X^2$, $X_3 = X_2 X$, $X_{i+1} = X_i X$. Similarly, set ${}_{i+1}X = X({}_i X)$. Take $x'_1, \dots, x'_n$ to be the co-factors of the elements in the first row of $\Delta'(X)$. Multiply them by the respective elements of the k -th row of $\Delta'(X)$ and add. We obtain the coefficient y_k of $Y = X'X$. Hence $y_1 = \Delta'$, $y_k = 0$ ($k > 1$). Thus $X'X = \Delta'$. Replace x_1 by $x_1 - \rho$ and let X' become

$$F = \sum_{i=0}^{n-1} F_i \rho^i,$$

where F_i are numbers of the algebra. Thus

$$F(X - \rho) = \sum_{i=0}^n l_i \rho^i.$$

By equating coefficients of like powers of ρ , we get

$$F_0 X = l_0, \quad F_1 X - F_0 = l_1, \quad F_2 X - F_1 = l_2, \quad \dots, \\ F_{n-1} X - F_{n-2} = l_{n-1}, \quad -F_{n-1} = l_n.$$

Multiply the second on the right by X , the third by X twice, the fourth by X three times, etc. Adding we get

$$\sum_{i=0}^n l_i X_i = 0.$$

Hence the left-hand characteristic determinant of X vanishes if we replace ρ^i by X_i (i. e., it has X as a right-hand root). Similarly, any element X is a left-hand root of the right-hand characteristic determinant.

Various properties of polynomials in ρ have their analogue in those of the linear combinations of the right hand powers X_i (or the left hand powers of X). If

$$A_0 = \sum_{i=0}^n a_i X_i,$$

we set

$$A_k = \sum_{i=0}^n a_i X_{i+k},$$

whence $A_k = A_{k-1}X$. Then if $\sum a_i \rho^i$ and $\sum b_i \rho^i$ have the greatest common divisor $\sum g_i \rho^i$, we have $A_0 = \sum m_i G_i$, $B_0 = \sum n_i G_i$, while $G_0 = \sum p_i A_i + \sum q_i B_i$, in which the m , n , p , q are scalars, $B_0 = \sum b_i X_i$, $G_0 = \sum g_i X_i$. In particular, A_0 and B_0 both vanish if and only if G_0 vanishes. Various other results follow in the usual manner.

4. Every prime ideal P in an arbitrary algebraic number field admits, as is well known, of primitive roots. But for P^n , where $n > 1$, primitive roots do not always exist. In what cases primitive roots of P^n exist was determined by A. Wiman in 1899.* It is the object of Professor Westlund's paper to treat the somewhat more general problem of actually determining the exponent, mod P^n , to which any primitive root of P belongs.

* Öfversigt af kongliga vetenskapsakademiens Förhandlingar, vol. 56.

5. Studies in the continuum perhaps too often ignore Hilbert's pregnant remark that the totality of real numbers has not yet been defined. Any scheme for producing transfinite series of fundamental sequences is nothing more nor less than a process of defining new sets of irrational numbers. It is a problem of epistemology whether the human intellect can ever exhaust the possibilities of such definition. Dr. Frizell considers the following example.

Let us define inductively a function of $2, 3, \dots, n$ arguments

$$e(\alpha, \beta) = \alpha^\beta, \quad e(\alpha_{i+1}, \alpha_i, \dots, \alpha_1) = e\{\alpha_{i+1}, e(\alpha_i, \dots, \alpha_1)\} \\ (i = 1, 2, \dots, n-1),$$

for example,

$$e(2, 3, 5) = 2^{2^5}.$$

First let the values of α be the well ordered set of positive fractional numbers (integers included, except unity). Next let each α take in succession all the above values of the e function and repeat the procedure indefinitely.

Now define a new function f such that for integral values of n $f(\alpha, n) = e(\alpha, \alpha, \dots, n \text{ arguments})$, e. g., $f(2, 5) = 2^{65536}$. If we succeed in defining $f(\alpha, \beta)$ for all values of α, β in the set defined above by the e function, there will follow the problem of defining in like manner a function f_1 so that $f_1(\alpha, n) = f(\alpha, \alpha, \dots, (n \text{ arguments}))$, and so on. But we cannot define all such functions.

6. In Professor Cajori's paper it is shown that what, since the time of Fourier, has been called "Newton's method" of approximation to the roots of numerical equations is not Newton's method, but Raphson's modification of it. Before 1800 the methods of Newton and Raphson were almost invariably kept distinct. Professor Cajori suggests that the approximation, as described in modern texts, be called the "Newton-Raphson method."

This paper will appear in full in the February number of the *American Mathematical Monthly*.

7. In most treatments of one-parameter families and nets of plane curves, the interest is concentrated altogether upon the determination and discussion of its envelope. More recently some authors, especially Lilienthal and Scheffers, have discussed a number of other interesting problems associated with such

families of curves, but all of these are of a metric nature. In Professor Wilczynski's paper the point of view of projective differential geometry has been adopted, the analytical treatment being based upon the invariant theory of a completely integrable system of partial differential equations of the form

$$\begin{aligned}y_{uu} &= ay_u + by_v + cy, \\ y_{uv} &= a'y_u + b'y_v + c'y, \\ y_{vv} &= a''y_u + b''y_v + c''y.\end{aligned}$$

The second equation of this system is what Darboux calls a Laplacian equation, and there exists an extensive theory concerned primarily with the cases in which it may be integrated by quadratures. This theory here finds a new and simple interpretation. Moreover, the fact that this equation appears here merely as one of three, instead of by itself, adds deeply to the significance of the theory.

Other points discussed in the paper are the determination of a complete set of invariants and covariants, the osculating conics of the individual curves of the net and their mutual relations.

8. In this paper Dr. Burgess shows that the set of circular numbers defined below are arithmetical invariants of a plane curve under the transformations of the circular group G_c . The small letters are Plücker's well-known numbers.

M : The degree of C is the number of points of C in common with an arbitrary circle. For a circular curve, $M = \mu$.

R : The class of C is the number of circles of an arbitrary pencil touching C . For a circular curve, $R = \nu$.

N : The inflexion of C is the number of osculating circles of C through an arbitrary point. For a circular curve, $N = i$.

A : The number of stationary circles of C . The same for a circular curve.

B : The cuspidal equivalence of C . For a circular curve, $B = \kappa$.

X : The number of doubly orthogonal circles of C through an arbitrary point. For a circular curve, X = the number of binormals.

Y : The number of bitangent circles of C through an arbitrary point. For a circular curve, $Y = \tau$.

G : The number of pairs of osculating circles of C which intersect upon an arbitrary circle. For a circular curve, G is the number of pairs of osculating circles having a common radical axis.

H : The number of pairs of tangent circles of C belonging to an arbitrary net while each circle of the pair passes through the point of contact of the other. For a circular curve all the circles may have their centers upon an arbitrary straight line.

D : The nodal equivalence. For a circular curve, $D + H = \delta$.

I : The number of intersections of C with a minimum line through I , excluding the point I .

J : The number of intersections of C with a minimum line through J , excluding the point J .

F : The number of foci of C , $F = (R - 2I)(R - 2J)$.

9. In his second paper Dr. Burgess proves the following theorem: The necessary and sufficient condition that the anharmonic curves upon the quadric $Q \equiv \sum a_{ik} x_i x_k = 0$, $|a_{ik}| \neq 0$, whose tangents belong to the null system $N \equiv \sum b_{ik} x_i x_k = 0$ may be rational skew curves of degree n , when the matrix $\|\lambda a_{ik} + b_{ik}\|$ has the characteristic [1111], is that the two positive roots α and β of the equation $|\lambda a_{ik} + b_{ik}| = 0$ be in the ratio $(n - 2) : n$.

Proof: The canonical forms in the case [1111] may be taken as $Q \equiv x_1 x_4 + x_2 x_3 = 0$, $N \equiv \alpha p_{14} + \beta p_{23} = 0$; and Q has the parametric equations $X_1 : X_2 : X_3 : X_4 = uv : u : v : -1$. Determining from these $\rho p_{14} = X_1 dx_4 - X_4 dx_1 = u dv + v du$ and $\rho p_{23} = X_3 dx_2 - x_2 dx_3 = u dv - v du$, he finds the differential equation of the curves to be $\alpha(u dv + v du) + \beta(u dv - v du) = 0$.

On integrating, the curves are $u = cv^{\frac{\beta + \alpha}{\beta - \alpha}}$, whence in the parameter v the curves are given by

$$\rho X_1 = cv^{\frac{\beta + \alpha}{\beta - \alpha} + 1}, \quad \rho X_2 = cv^{\frac{\beta + \alpha}{\beta - \alpha}}, \quad \rho X_3 v, \quad \rho X_4 = -1.$$

Obviously the necessary and sufficient condition that the curves be rational and of degree n is

$$\frac{\beta + \alpha}{\beta - \alpha} + 1 = n, \text{ whence } \frac{\alpha}{\beta} = \frac{n - 2}{n}.$$

In like manner, he finds in the other canonical forms that the cases [(11)11], [(11)(11)], [(22)] and [(31)] are rational and of degrees 2, 1, 1, and 2 respectively. The one remaining form [22] has the curves

$$X_1 : X_2 : X_3 : X_4 = cu e^{2\alpha u} : u : ce^{2\alpha u} : -1,$$

and these are not rational for any value of the root α .

10. In his third paper, Dr. Burgess derives the following method for carrying out a linear substitution with matrix H upon a cubic form $\sum a_{ijk}x_ix_jx_k = 0$ ($i, j, k = 1, 2, \dots, n$).

Consider the cubical matrix $\|a_{ijk}\|$ and build from it the series of plane matrices $A_q \equiv \|a_{iqj}\|$ ($q=1, 2, \dots, n$). Compute the series of matrix products $H'A_qH$ ($q=1, 2, \dots, n$), H' being the conjugate of H . From this series, build the new series $B_p \equiv \|b_{ipk}\|$ ($p=1, 2, \dots, n$) by writing for the i th column of B_p the p th column of $H'A_rH$ ($r=1, 2, \dots, n$). Compute from this last series the set of matrix products $B_pH \equiv \|c_{ipk}\|$ ($p=1, 2, \dots, n$); and construct from these the cubical matrix $\|c_{ijk}\|$. This is the cubical matrix of the transformed cubic form, $\sum C_{ijk}x'_ix'_jx'_k = 0$.

11. Professors Risley and MacDonald discuss the conditions under which a one-parameter family of plane curves will possess an envelope. The discussion is exhaustive for the curves $y=f(x, \alpha)$, and partial for the curves $F(x, y, \alpha)=0$. The treatment is applicable whether the variables are real or complex.

The authors obtain a necessary and sufficient condition that

$$P(x, y) \equiv y^n + \sum_{i=1}^n A_i(x) \cdot y^{n-1}$$

be irreducible at $(x_0, y_0=0)$. They then use this theorem and Weierstrass's implicit function theorem to establish a theorem which summarizes and exhausts the facts concerning envelopes of the curves $y=f(x, \alpha)$.

For the curves $F(x, y, \alpha)$, the authors consider two cases: (a) that in which the curves have no multiple points in the region in question, (b) that in which the curves have these multiple points of order two only. They obtain sufficient conditions for an envelope in case (a). Under case (b) they obtain sufficient conditions for the existence of a nodal locus, and conditions necessary and sufficient that a locus of double points be an envelope locus.

12. Let us say that a power series $f(x_1, \dots, x_n; z_1, \dots, z_p)$ is of order m in $x_1, \dots, x_n$ provided the terms of lowest degree in $f(x_1, \dots, x_n; 0, \dots, 0)$ are of degree m in $x_1, \dots, x_n$.

There is a theorem of Weierstrass to the effect that a power series $f(x; z_1, \dots, z_p)$ of order n in x can be factored in the

form $f(x; z_1, \dots, z_p) = [x^n + a_1 x^{n-1} + \dots + a_n] \cdot g(x; z_1, \dots, z_p)$, where $g(x; z_1, \dots, z_p)$ is a power series in $x; z_1, \dots, z_p$ with a constant term distinct from zero, and $a_1, \dots, a_n$ are power series in $z_1, \dots, z_p$ vanishing for $z_1 = \dots = z_p = 0$. Since g does not vanish for $x = z_1 = \dots = z_p = 0$ it follows that for solutions in the neighborhood of the origin the equations $f = 0$ and $x^n + a_1 x^{n-1} + \dots + a_n = 0$ are equivalent.

It is the purpose of Dr. MacMillan's paper to show that if $f_1(x, y; z_1, \dots, z_p)$ and $f_2(x, y; z_1, \dots, z_p)$ are two power series of order m and n respectively in x and y and if a certain eliminant E is distinct from zero, then two polynomials $p_1(x, y)$ and $p_2(x, y)$ whose coefficients are convergent power series in $z_1, \dots, z_p$, can be found such that in the neighborhood of the origin the system of equations $p_1 = p_2 = 0$ admits the same solutions for x and y as functions of $z_1, \dots, z_p$ as does the system $f_1 = f_2 = 0$, and therefore under the hypothesis that $E \neq 0$ the system $f_1 = f_2 = 0$ admits mn solutions for x and y as functions of $z_1, \dots, z_p$ which vanish for $z_1 = \dots = z_p = 0$. These results can be used to show that three power series in three dependent variables of order l, m , and n respectively admit in general lmn solutions in the neighborhood of the origin.

13. Dr. MacNeish's paper considers the determination by means of the calculus of variations of the path of light in a medium homogeneous in concentric spherical layers. The question of conjugate points for the various types of paths is especially considered. When the absolute index of refraction n is a continuous function of the position, the path of any ray in the medium is such as to make the integral

$$I = \int_{s_1}^{s_2} n ds,$$

where s is the length of the path, a minimum or a maximum according to the curvature of the refracting surface. If the medium is homogeneous in concentric spherical layers, the path minimizes the integral I . In this case the path lies in a plane through the center of the spheres. The problem is taken up in the plane of the path and six types of paths are obtained, designated types A, B, C, D, E, F , of which only paths of type E may have conjugate points. A path of type E ascends and reaches a maximum and then descends symmetrically to the level of the initial point. If the amplitude of the maximum

point increases with the initial horizontal angle of the path, there are no conjugate points on the path. If the amplitude of the maximum point decreases as the initial horizontal angle of the path increases, the maximum point is a conjugate point for the path.

14. The paper of Professor Laves is a continuation of a former paper published by him in the *Astronomische Nachrichten*. Besides giving a simple geometrical interpretation of Haeckel's theorem concerning the integration of Jacobi's partial differential equation for conservative systems, it takes up certain analytical results of Jacobi for elliptical coordinates and shows how they can be simplified by the same geometric conception.

15. The first systematic exposition of algebra was doubtless that written in the early part of the ninth century by Mohammed ibn Musa al-Khowārazmī. Translations from the Arabic into Latin were made in the first half of the twelfth century. One version, probably by Gerhard of Cremona, has been published by Libri in his *Histoire des Sciences mathématiques en Italie*. An independent version by an Englishman, Robert of Chester — probably also known as Robert of Reading — exists in manuscript in Vienna, in Dresden, and a third copy in the Columbia University library. Professor Karpinski presents a report on the content of this version, showing also that the two Latin versions are doubtless from the same Arabic original of which Frederic Rosen in 1830 published the text and English translation. While several references to the work by Robert of Chester have appeared, no other study of the content has been made.

16. The question of the origin of the Hindu-Arabic numerals is complicated by the fact that two forms of numerals with place value were used by Arabic writers of the tenth and succeeding centuries. Of these the gobār or dust forms resemble more nearly the ones which we use, while the so-called Hindu forms are more like the ones used by the modern Arabs. Sometimes, with both forms, superposed dots were employed instead of zeros to indicate multiples of the powers of ten, as 5̇ for 5000. Professor Karpinski has found in the *Kitāb al-Fihrist* an entirely new and hitherto uncited usage of subscript dots with the same force as the superposed dots.

The reference is of further importance as being one of the earliest (987 A.D.) discussions of the origin of the numerals.

17. Defining a ruled surface in four-dimensional space as a two-dimensional continuum made up of ∞^1 lines, and a planed hypersurface as a three-dimensional continuum made up of ∞^1 planes, Dr. Ranum derives some of the projective properties of these correlative loci. For instance, he shows that every non-developable ruled surface R lies on one and only one developable planed hypersurface P . Given P , an infinite number of R 's exist, arranged in pairs of conjugates such that every generator of one meets three consecutive generators of the other; any two conjugates meet in a curve whose tangent surface (developable) also lies on P . Dually, every non-developable planed hypersurface H contains one and only one developable surface S . Given S , an infinite number of H 's exist, arranged in pairs of conjugates such that every generating plane of one meets three consecutive generating planes of the other in straight lines; the envelope of the common tangent hyperplanes of any two conjugates is a developable hypersurface, which also contains S .

18. Dr. Neikirk's paper generalizes and extends the results published by him in the *Transactions* for July, 1910. It is shown in the first part that every transformation group defines a substitution group of finite or infinite degree. The problem of constructing the transformation, having given the substitution, is considered and solved in certain cases. The second part is devoted to the discussion of the properties of substitutions of an infinite degree.

19. In Professor Shaw's paper, the advantage of introducing into the analytic treatment of geometry of a surface the differential operator δ , which is a two-dimensional variable Hamiltonian operator analogous to a variable ∇ , is shown. The operator δ is defined thus: Let $\rho = \phi(u_1, u_2)$, then $d\rho = \rho_1 du_1 + \rho_2 du_2$. The normal is $\nu = V \cdot \rho_1 \rho_2$. Then

$$\delta = \frac{1}{T\nu} \left(\rho_2 \frac{\partial}{\partial u_1} - \rho_1 \frac{\partial}{\partial u_2} \right).$$

If $\psi(u_1, u_2) = c$ is the equation of a curve on the surface, its tangent is $\delta\psi$. In any direction α tangent to the surface, the differential of ψ is $d_\alpha\psi = T\nu\alpha\delta\psi$. Evidently $d_\alpha\psi$ is a maxi-

imum for $\alpha \perp \delta\psi$. If ψ is called a "potential" function for the surface, $\psi = c$ gives the lines of equipotential and $\nabla\psi$ the lines of flow.

The differential parameters $\Delta_1(\psi)$, $\Delta_1(\psi, \theta)$, $\Theta(\psi, \theta)$ are

$$\Delta_1(\psi) = -(\delta\psi)^2, \quad \Delta_1(\psi, \theta) = -S\delta\psi\delta\theta, \quad \Theta(\psi, \theta) = TV\delta\psi\delta\theta.$$

The differential parameter $\Delta_2(\psi) = -S \cdot \delta^2 \psi$, where $\delta^2 \psi = \delta(\delta\psi)$, remembering that δ is to be differentiated also. The geodetic curvature is $1/\rho_g = S \cdot \delta U \delta\psi$. Around any area on the surface we have

$$\int d\rho g = \int \int \delta \cdot g.$$

We may thus have a divergence and a curl.

The method may be at once extended to space of three dimensions considered as lying in one of four; or to one of n dimensions lying in one of $n+1$, or to one of m lying in one of n . The symbolic forms for differential parameters are then seen to be these same formulas in a different notation.

20. In his dissertation Jüstus Grassman claims to have established the following theorem:

Für die Wendepunkte einer allgemeinen Curve m^{ter} Ordnung gilt der folgende Satz: Legt man durch $\frac{1}{2}(m-2)(m+1)$ Wendepunkte einer Curve m^{ter} Ordnung eine Curve $(m-2)^{\text{ter}}$ Ordnung hindurch, so schneidet diese die erstere in $\frac{1}{2}(m-1)(m-2)$ Punkten, die gleichfalls Wendepunkte sind.

It is Dr. Slobin's intention to show that the theorem is not generally true. He applies the theorem to non-singular quintic curves having 15, 11, 7, 3 real inflexions, whose existence has been demonstrated by Barcroft, Klein, and others. With a proper choice of the $\frac{1}{2}(m-2)(m+1)$ inflexions as a base of a sheaf of curves of order $(m-2)$ the theorem leads to a contradiction of Plücker's formulas. We may apply the theorem to curves of higher orders, and it will be found to lead to an impossibility in many cases. The paper also takes up the classification of quintic curves as to their branches, bitangents, and multiple points.

21. This paper treats the case of two finite bodies describing elliptical orbits, and an infinitesimal body moving in the vicinity of the Lagrangian solution points. Professor Moulton discusses the properties of the orbits which are periodic in both two and three dimensions, and shows how to construct the periodic series for their coefficients.

22. The primary object of Professor Williams's paper is to prove, for scrolls of order 5, Professor Story's formula for the number of intersections of any two curves lying on the same ruled surface, i. e.,

$$[C_{a,\alpha}C_{b,\beta}] \equiv a\alpha + b\beta - \mu\alpha\beta,$$

where $C_{a,\alpha}$ is a curve of order a that meets each generator of the ruled surface R_μ of order μ in α points, α being called the grade of the curve, and $C_{b,\beta}$ is a curve of order b and grade β on R_μ . If the curves pass through the same point of a multiple curve on the ruled surface, they are not to be regarded as intersecting there unless they lie on the same sheet. For two curves on R_μ , of multiplicities m and n respectively, say $C_{a\cdot m\alpha}$ and $C_{b\cdot n\beta}$ the formula is

$$[C_{a\cdot m\alpha}C_{b\cdot n\beta}] \equiv m\alpha\beta + nb\alpha - \mu\alpha\beta.$$

It is proved that on any scroll R_μ having a director line of multiplicity $\mu - 1$ the number of intersections of $C_{a,\alpha}$ and $C_{b,\beta}$ is equal to the order of the residual intersection of the two cones of orders a and b respectively through these curves from an arbitrary point on the multiple director as a vertex. If S_ν is a surface of order ν , used to cut a curve $C_{b,\beta}$ from the scroll, $\frac{5}{2}\nu(\nu - 1) + 4$ points for the determination of S_ν can be arbitrarily chosen and S_ν will not break up into R_b and $S_{\nu-b}$. S_ν must be chosen so that either the residual intersection shall consist entirely of generators or of generators and a curve of order less than b . The following values of ν were found:

$\nu = b$ and S_ν a cone for scroll I (Schwarz's classification, *Orelle*, volume 67).

$\nu = \frac{1}{5}(2b + 8 + n)$, ($n = 0, 1, 2, 3$ or 4) for scrolls III, IV and V.

$\nu = \frac{1}{5}(2b + 11 + n)$, ($n = 1, 2, 3, 4$ or 5) for scrolls II, VI, VII, VIII, IX, X.

$\nu = \frac{1}{5}(2b + 13 + n)$, ($n = 0, 1, 2, 3$ or 4) for scrolls XI and XIV.

$\nu = \frac{1}{5}(2b + 6 + n)$, ($n = 0$ or $5, 1, 2, 3, 4$) for scrolls XII and XIII.

And for scroll XV, for $\beta = 1$, $\nu \geq b - 1$; for $\beta = 2$, $\nu \geq 2b - 3$; for $\beta \geq 3$, $\nu \geq \frac{1}{2}[1 + (2b + 1)\beta - 5\beta^2]$.

H. E. SLAUGHT,
Secretary of the Section.

PREPARATION FOR RESEARCH AND THE
DOCTOR'S DEGREE IN MATHEMATICS.REPORT OF THE AMERICAN SUBCOMMITTEE OF THE INTERNATIONAL COMMISSION ON THE TEACHING OF
MATHEMATICS.*

THE following institutions are mentioned in this report:

Brown,† Bryn Mawr, California,† Chicago, Cincinnati, Clark, Columbia, Cornell, Harvard, Illinois, Indiana, Johns Hopkins, Kansas, Michigan, Minnesota, Missouri, Nebraska, Northwestern, Pennsylvania, Princeton, Stanford, Syracuse, Virginia, Wisconsin, Yale.

The requirements for the degree of doctor of philosophy in mathematics are not exactly defined in all the above institutions. As a matter of fact, the policy is in force in some of them of definitely advising aspirants for the doctorate to complete their graduate studies at other universities in which graduate faculties in mathematics have long been strong and active. It is, however, clear that this condition is but temporary. The increase in the number of strong men in mathematics is resulting in added strength in an increasing number of institutions. It is, therefore, certain that the number of universities in which graduate work will be carried to a definite conclusion will be greater in the immediate future than at present.

On the other hand, in those universities in which within the past ten years three or more doctorates have been awarded there is a degree of uniformity in the requirements which probably indicates and establishes a standard for the United States—the outgrowth of conditions peculiar to the educational system of this country. This standard may be described by detailing the course and career of a successful candidate in some one of these institutions.

For admission to registration as a candidate for the doctorate, successful completion of an undergraduate course terminating

* *International Commission on the Teaching of Mathematics.* Committee XIV.—Graduate Work in Universities. Subcommittee 2—Preparation for Research and the Doctor's Degree. Chairman, P. F. SMITH, Yale University; MAX MASON, University of Wisconsin; E. J. WILCZYNSKI, University of Chicago; J. I. HUTCHINSON, Cornell University; G. D. BIRKHOFF, Princeton University. This report was submitted to the Commissioners in December, 1910.

† Data taken from catalogues and necessarily incomplete.

with a bachelor's degree is required. While all undergraduate degrees cannot at the present time be regarded as equivalent in educational units, greater uniformity in the future will result from the standardizing influence in this direction of the Carnegie Foundation for the Advancement of Teaching.

In his undergraduate years, the candidate has pursued courses in mathematics including the calculus, and in some instances also mechanics, projective geometry, advanced calculus, and theory of functions. In any case the courses mentioned are regarded as introductory, and are required before advanced study begins. A reading knowledge of French and German is prescribed. The course of advanced study upon which the candidate now enters extends usually over three years. This time may, however, be spent in residence at one or more universities, with the exception of the final third year. During this year, the candidate must reside at the university from which he desires his academic degree. The instruction during the period mentioned is for the most part by lectures combined with work in the seminary prepared and delivered by the candidate and his fellow students under the critical eyes of their professors. There may be no formal examinations during this period. The responsibility of mastering his courses rests upon the candidate. In addition to mathematics, one or two minor subjects are required, usually in applied mathematics. This portion of the requirement may, in some cases, be successfully met before the final year.

The mathematical club affords an opportunity for presentation of original papers and discussions of pedagogical and pertinent questions. In the mathematical seminary library are available the standard treatises for study and consultation in connection with the lectures.

The final test of the candidate's power in original investigation is the quality of his thesis. If this is satisfactory, and if he survives the ordeal of a final examination planned to determine the extent of his mathematical attainments, he must, as a final requirement, arrange for the publication of his thesis, in whole or in part, in a manner satisfactory to the department under which he has studied. A small diploma fee is required when the degree is granted.

A summary of the existing conditions at the universities named above follows, the arrangement being topical for the sake of clearness.

Conditions of Matriculation.

The bachelor's degree accepted in matriculation is not specifically described at the following institutions, but all such degrees are considered equivalent: Cincinnati, Illinois, Missouri, Stanford, Syracuse, Virginia, Yale.

"Graduates of this or other institutions in which the bachelor's degree is awarded for courses equivalent to those given at this institution" are accepted at Bryn Mawr, Chicago, Cornell, Kansas, Nebraska, Wisconsin.

A bachelor's degree from a college "in good standing" is required by Brown, California, Clark, Columbia, Johns Hopkins, Princeton, Pennsylvania.

Harvard University requires a bachelor's degree of "substantial character."

At Michigan the credentials of the candidates are examined by the Graduate Council.

It appears that, at the majority of the institutions in question (16 out of 24), the mere attainment of the first degree is not deemed sufficient, but the value of the degree in educational units is decisive.

Language Requirement.

All institutions prescribe a "good reading knowledge" of French and German. The practical point is that the candidate may have to consult treatises and memoirs in French, German, or Italian, and his linguistic knowledge must suffice for this purpose. In addition, Bryn Mawr specifies "some knowledge" of Latin as a requirement, and at Brown and Yale a "good knowledge" of Latin is required unless for some excellent reason the candidate is excused by the faculty.

Time Element.

One year of continuous residence is required in all cases save by Bryn Mawr (2 years), Chicago ($1\frac{1}{2}$ years), Columbia (2 years).

Work of equal grade in other universities is generally credited for courses in the early graduate years.

It appears, as a rule, that three years of graduate study are necessary for securing the doctorate.

Prescribed Courses — Number of Courses.

Rigidly prescribed graduate courses in mathematics do not, as a rule, exist. Specific exceptions are the following:

Courses totalling 20 hours for one year at Clark University. The titles of the courses are analytic geometry of higher plane curves, skew curves and surfaces, differential equations and calculus of variations, real and complex function theory including elliptic functions, theory of numbers, modern synthetic geometry, theory of substitutions, invariants, finite differences.

Undergraduate courses amounting to 8 hours for a year at Yale University. The subjects are projective geometry, advanced calculus, mechanics, function theory.

The courses just mentioned are doubtless required at all institutions in which degrees have been awarded, without specific mention of this fact.

The amount of work required is also not generally defined. There are some exceptions. At Yale, a candidate must have successfully followed graduate courses amounting to 18 hours for a year. At Columbia, a minimum of 15 hours is required; at Virginia, 9 hours of lectures each year; at Pennsylvania, 24 year-hours.

The essential point is, of course, that the candidate must prepare himself for the final examination. To make a comparison of the exacting nature of this work in various institutions is difficult and perhaps impossible. The same course (in name) might well be quite vigorously conducted at one university, and at another partake of an easy character. On the other hand, the *breadth* of study required can easily be determined by taking into consideration minor subjects (see the next section).

Minors.

The requirement as to minor subjects at Harvard University is quite definite. From the following four divisions of mathematics: I. Geometry, II. Arithmetic and algebra, III. The differential and integral calculus and the theory of functions, IV. Mechanics and mathematical physics, the candidate must designate one as his chosen field, his major. In each of two of the other three, he must present a thesis on a topic assigned to him. "These minor theses are intended to test the capacity of the candidate to prepare, at short notice, brief expositions of subjects with which he has had previously little or no acquaintance. While no strictly original work is expected on these theses, the form of presentation must be original."

Minor subjects (one or two) are as a rule required, but not specified. Princeton offers an exception in specifying philosophy

and a cognate subject. At Wisconsin, one minor must be studied for two years, a second minor for one year.

At Yale, the requirement of 18 year-hours of graduate courses results in general in the pursuance of courses distributed among three of the main divisions in the Harvard scheme.

Mathematical physics is very generally recommended as a minor.

Final Examination.

Examinations in course are held by Columbia, Harvard, Illinois, Kansas, Missouri, Northwestern, Pennsylvania, Syracuse, Virginia, Wisconsin, Yale.

A formal, final oral examination is required after acceptance of the thesis at the following universities: Chicago, Clark, Columbia, Harvard, Minnesota, Nebraska, Northwestern, Princeton, Wisconsin.

The final examination may be both oral and written at these institutions: Bryn Mawr, California, Cincinnati, Cornell, Illinois, Johns Hopkins, Kansas, Michigan, Missouri, Stanford, Syracuse, Yale.

Yale offers an exception to the general rule of holding a final examination at the end of the candidate's course. At Yale, the examination takes place at the end of the last year but one, or at the beginning of the last year. This determination of the candidate's attainments before work on the thesis is begun is intended as a check.

The final examination is public except at Clark, Cornell, Minnesota, Pennsylvania, and Yale.

Virginia requires no final examination; Pennsylvania, a written examination.

Publication of Thesis.

Publication of the thesis, in whole or in part, is a requirement at all institutions (page 1) except Brown, Harvard, Michigan, Minnesota.

Special conditions imposed are:

There must be deposited in the university library: 150 copies (Bryn Mawr), 25 copies (Northwestern), 100 copies (Clark); 100 copies (Chicago, Cincinnati, Illinois, Nebraska, Virginia, Wisconsin), 50 copies (Cornell), 30 copies (Columbia).

Printing of the entire thesis is required by Bryn Mawr, Chicago, Cincinnati, Columbia, Indiana, Northwestern, Pennsylvania, Princeton, Syracuse, Virginia.

Acceptance by a reputable journal of the whole or part suf-

fices at Clark, Cornell, Johns Hopkins, Indiana, Illinois, Michigan, Missouri, Nebraska, Pennsylvania, Yale.

At Harvard, doctors' dissertations have "almost invariably been published."

Various.

A small diploma fee is usually required. This varies from five dollars to sixty dollars (Princeton).

The master's degree is in no case a prerequisite for the doctorate, but is recommended at Chicago, Cincinnati, Columbia, Illinois, Kansas, Minnesota, Missouri.

Research and Special Courses.

The candidate's ability for original investigation is, in general, stimulated through work in seminars. Topics assigned to him for private research develop this faculty, and criticism and personal conference with his instructors further the same end. Seminary work consists in discussion and criticism of original papers presented by candidates. Reports on current literature are also called for, in particular on contributions to those special lines in which students and instructors are peculiarly interested.

Equipment—Special Features.

Collections of mathematical models are available, in all institutions, for study by the candidate. At Yale, a special room is equipped with all necessary tools for the making, by the student, of simple models. With the development of graduate clubs at universities throughout the United States has come the establishment in each institution of a mathematical club. At Harvard the club is conducted entirely by the students, members of the faculty attending as guests. The organization at Yale differs from this only in the existence of faculty supervision through the secretary of the club, who is a full professor. This supervision is nominal. In general it may be said that the mathematical club is an informal affair, professors and students meeting for extra-curriculum intercourse and for discussion of such papers and reports as may be presented by faculty and students without obligation so to do.

The library facilities are reported as adequate for all lines of mathematical research, save at Indiana, Kansas, Michigan, Minnesota, Syracuse, and Wisconsin. In addition to the resources of the university library there exists at many institutions a seminary library in which are available current mathematical treatises and periodicals—duplicates of those stored in the main library.

The library of the American Mathematical Society is rapidly becoming for its members a most valuable possession. By exchange, files of all periodicals and proceedings are being completed and these are made available to members wherever located through the regular means of transportation.

Degrees Granted.

Within the years 1900-09, the doctorate in mathematics has been conferred upon approximately 135 candidates, the number at the various institutions being as indicated (in a few cases round numbers are given):

Bryn Mawr 1, California 1, Chicago 30, Clark 9, Columbia 10, Cornell 14, Harvard 12, Illinois 1, Johns Hopkins 20, Kansas 1, Pennsylvania 8, Princeton 3, Stanford 1, Syracuse 1, Virginia 4, Wisconsin 4, Yale 21.

Conditions in the Faculties.

At Clark, Chicago, and Johns Hopkins certain members of the mathematical faculties give graduate instruction only. The number of hours per week of such instruction is respectively: Clark 7, Chicago 6-8, Johns Hopkins 4-7.

In general, graduate instruction is compensated for by a reduction in undergraduate hours. The following table makes clear the policy followed in the various institutions, the first number indicating the hours per week of undergraduate, the second of graduate instruction:

Bryn Mawr	9	and 3
Chicago	6	" 4
Columbia	6	" 6 or 9 and 3
Cornell	11	" 4
Harvard	4½	" 4½ and a reading course
Illinois	10	" 3
Indiana	5	" 4 or 5 and 6
Kansas	7	" 5 or 10 and 3, or 15 and 0
Michigan	12	" 3
Minnesota	11	" 3
Missouri	10	" 2 or 10 and 3
Nebraska	10	" 3 or 15 " 0
Northwestern	6	" 6 or 10 " 3
Pennsylvania	10	" 3
Princeton	6	" 3 or 9 and 3
Stanford	10	" 0
Syracuse	6	" 6
Virginia	12	" 2
Yale	10	" 3

SHORTER NOTICES.

Theories of Parallelism, an historical Critique. By W. B. FRANKLAND. Cambridge, 1910. xviii + 70 pp.

AFTER reading *The Story of Euclid* (1901), and particularly after reading *Euclid Book I with a Commentary* (1902), by the same author, every student of the history of mathematics will welcome this little "historical critique," feeling sure of finding much food for thought in small compass. For it is one of the characteristics of Mr. Frankland that he says what he has to say in the fewest words possible, and hence in a book of less than a hundred pages he condenses matter that most writers would expand to fill double the space. And when we come to consider that in these few pages he has presented a scholarly digest of the theories of upward of forty geometers we begin to realize the thought that has been given to the subject and the skill that the author has shown.

Mr. Frankland gives a setting for the historical discussion in an introduction of eighteen pages. In this he begins by stating one existence assumption and one fundamental theorem as follows: "Let us assume that *straight lines* are freely applicable to themselves and to one another; and that there is a *plane* in which they are freely movable; and let us investigate the *parallelism* of such straight lines in an even plane." The theorem is that of Hilbert, that the area of any polygon is proportional to its divergence, that is, to the difference between its angle sum and $(n - 2)\pi$. From this theorem, of which the usual proof is given, the author proceeds to prove that for the elliptic, parabolic, and hyperbolic hypotheses, respectively, there are no, one, and two parallels to a given straight line, through any given point.

The historical sketch is then introduced by a discussion of Euclid's own theory, a discussion that has, of course, been anticipated by Dr. (now Sir Thomas) Heath's monumental work on the *Elements*. Heath calls attention to the fact that Euclid assumes the infinitude of space, and that the possibility of a straight line as re-entrant seems never to have occurred to him. He thus bars from his theory the possibility of the non-existence of parallelism. On the other hand the fifth postulate bars the possibility of double parallelism, so that there is left to him only the parabolic hypothesis. The weakest feature of his theory is that the statement of the fifth postulate positively invites attempts

at proof, and, as we know, these attempts were constantly being made for over two thousand years.

Of the successors of Euclid, Posidonius (B. C. 80) used the equidistant definition, thus barring out the elliptic and hyperbolic hypotheses. Geminus (B. C. 70) has often been made to bear the blame of the definition of one Aganis, that equidistants "are such as lie in one surface, and when produced indefinitely have one space between them, and it is the least line between them." Mr. Frankland follows Dr. Heath in believing this Aganis to have been a writer about A. D. 500. Ptolemy (A. D. 150) wrote a tractate on the fifth postulate, giving four propositions by way of proof. Of these the first is substantially as follows: "If two straight lines are crossed by a transversal so that the interior angles on the same side are together equal to two right angles, then the lines can never intersect." The proof consists in showing that if they meet at O they must also meet on the other side of the transversal at O' . The possibility that O and O' may be identical, as they are in the elliptic hypothesis, evidently did not occur to Ptolemy. Proclus (A. D. 450) seems to have been the first writer to have had any of the modern view of the nature of parallelism. As Mr. Frankland says, his words might almost have been written by Lobachevsky or Bolyai. Without entering into the geometric discussion, one passage is worthy of special note. Proclus says, "It cannot be asserted unconditionally that straight lines produced from less than two right angles do not meet. It is of course obvious that some do meet, but the (euclidean) theory would require all such to intersect. But it may be urged that as the defect from two right angles increases, the straight lines continue asecant up to a certain magnitude of the defect, and for a greater magnitude than this they intersect." It is the last phrase that is significant. The next noteworthy writer on the subject was Nasr-Eddin (A. D. 1250), whose attempt is set forth in the works of Wallis. His effort is well known, tacitly begging the question by another postulate as difficult as that of Euclid. Other writers of more or less prominence are Anaritius (A. D. 900), Gerbert (A. D. 1000), Billingsley, who edited the first English edition of Euclid (1570), Clavius (1574), Oliver (1604), Savile (1621), Tacquet (1654), and Hobbes (1655). In 1663 Wallis gave his proof, or rather his well-known substitute for the postulate. This substitute is assumed "as an universal idea: To any given figure whatever, another figure, similar and of any size, is possible." Leibnitz

(1679) seems to have had "logical premonition of the elliptic hypothesis" in the note that he gives on the definition of Euclid.

Coming to what may be called the modern school, the name of Saccheri (1733) naturally stands out as chronologically the first. His theory is too well known through the work of Engel and Stäckel to need any description in a review, but its prime weakness lay in the failure to consider the possible curvature of three-dimensional space. Simson's (1756) substitute for Euclid's postulate is well known as an educational rather than a mathematical effort. Lambert's (1766), however, is quite the reverse, and essentially he recognizes the three possible hypotheses which Klein finally named the parabolic, elliptic, and hyperbolic. Bertrand's (1778) theory is less familiar, and it has all the charm of style that characterized this writer, but it has not the breadth of view of Lambert's, nor indeed of Saccheri's. Playfair's (1795) adaptation of the proposition of Proclus is well known, since it is the postulate of parallels of our ordinary textbooks, and is noteworthy as an educational measure. The relations of Gauss to the Bolyais are fairly stated, and the hypothesis of the latter is set forth in a clear fashion. To Lobachevsky, however, Mr. Frankland, following Dr. Whitehead and others, gives the greatest praise, both in the matter of priority and of clearness. Such of the modern writers as have contributed to the theory, including Riemann, Cayley, Beltrami, Clifford, and Klein, are mentioned, thus bringing the "critique" up to the present time. Perhaps a quotation from Clifford, that may not have come to the attention of all readers, may be permitted, even at the risk of extending a review already too long: "I hold in fact: (1) That small portions of space are of a nature analogous to little hills on a surface which is on the average flat; namely, that the ordinary laws of geometry are not valid in them. (2) That this property of being curved or distorted is continually passed on from one portion of space to another after the manner of a wave. (3) That this variation of the curvature of space is what really happens in that phenomenon which we call the motion of matter whether ponderable or ethereal. (4) That in the physical world nothing else takes place but this variation, subject, possibly, to the law of continuity." As Frankland remarks, "the boldness of this speculation is surely unexcelled in the history of thought."

DAVID EUGENE SMITH.

Synthetische Theorie der Cliffordschen Parallelen und der linearen Linienörter des elliptischen Raumes. Von WOLFGANG VOGT. Leipzig und Berlin, Teubner, 1909. vii + 58 pp.

THE parallels of the hyperbolic space seemed for many years to furnish the only possible extension of the notion of parallelism derived from euclidean geometry. But about forty years ago the brilliant Clifford discovered in elliptic space straight lines which possess most of the properties of euclidean parallels, but differ from them in being skew. Two lines are right (or left) parallel if they cut the same right (or left) generators of the absolute. After Clifford's death, F. Klein and R. S. Ball made extensive contributions to the knowledge of the properties of these lines. More recently E. Study and J. L. Coolidge have been studying the subject. Analytic, synthetic, and differential geometry and vector analysis have been employed by various investigators. But the author of the present paper thinks that no *purely* synthetic treatment has appeared, although such a method is not only possible, but very suitable for the problem. He therefore aims to give the theory of the Clifford parallels in a purely synthetic form.

The article begins with an explanation of the notion of the "winding" of two lines. The Clifford parallels are defined as lines which have more than two common perpendiculars. The nets of parallels of both "windings" are deduced from the v. Staudt-Lüroth theory of line "nets" with imaginary directrices. At the same time, their projective connections with the absolute polar space and with each other are obtained. By elementary methods, it is shown that theorems on euclidean parallels hold true for the Clifford parallels. Other propositions due to modern writers are also demonstrated. The first part of the article is concluded by a brief discussion of the kinematics of elliptic space.

In the second chapter the three sections deal with ruled systems of the second order, the linear complex, and the linear congruence. The projective properties of these linear line loci are assumed and their metrical relations are deduced. Special attention is paid to the appearance of parallels. The parallel cone or the asymptotic cone of a ruled surface of the second order splits, in the elliptic space, into two cones coaxial with the surface. But two arbitrary coaxial cones do not form the parallel cones of a ruled surface of the second order unless certain conditions are fulfilled. A classification of ruled sur-

faces of the second order is made according to the form taken by their cones. It is interesting to compare this with the classifications of quadrics in elliptic space given by J. L. Coolidge (Non-Euclidean Geometry, page 156) and T. J. P'a. Bromwich ("The classification of quadric loci," *Transactions*, volume 6, 1905). In these articles the principles of classification are entirely different from that employed here.

In the section dealing with linear complexes, right and left complexes are distinguished, the existence of "diameter parallel nets" is proved, and the appearance of parallels in the linear complex and in the corresponding null space are investigated. Of special interest is the parallel complex, which possesses a whole net of axes and admits ∞^4 motions carrying it into itself, while the ordinary complex has only ∞^3 . The article is concluded by a discussion of the properties of the general linear congruence and some of its special forms.

E. B. COWLEY.

Das Gruppenschema für zufällige Ereignisse. Von HEINRICH BRUNS. Des XXIX Bandes der *Abhandlungen der Mathematisch-Physikalischen Klasse der Königl. Sächsischen Gesellschaft der Wissenschaften*, No. VIII. Leipzig, B. G. Teubner, 1906. Pp. 579-628.

THIS monograph is an extension of the brief development of the subject in the eighteenth lecture contained in the treatise by Bruns on *Wahrscheinlichkeitsrechnung und Kollektivmasslehre*. It is assumed in setting the simplest problem of the work that n balls are drawn, one at a time, from a bag containing balls of various colors, and that each time the ball drawn is returned to the bag before another drawing is made. The n consecutive drawings are called a draw series (Zugreihe) indicated by $Z(n)$. The draw series is written in the form

$$(1) \quad Z(n) = z_1 z_2 \cdots z_n,$$

where z_h denotes the h th drawing.

If the draw series are collected into sets of s with subscripts 1 to s , 2 to $s+1$, 3 to $s+2$, etc., the sets of s are called s -membered draw groups and the symbol $G(s)$ is used to designate such a group. In the formation of such groups, the author distinguishes between what he calls linear and cyclical groups. If, from (1), we take merely z_1 to z_s , z_s to z_{s+1} , $\cdots$, z_{n-s+1} to z_n ,

the formation is called linear, and the corresponding groups linear groups. If, in addition to these groups, we proceed to take sets of s by repeating z 's at the left of the draw series (1), the formation is called cyclical and the groups cyclical groups. The larger n is relative to s , the less significant is the difference of the two methods of group formation. It turns out that it is simpler to treat the cyclical groups than the linear groups, and it is of importance that the two do not differ significantly for large values of n .

Associated with each drawing Z_k , the author writes p_k , the probability of drawing that particular color. If in any group $G(s)$ we replace each z by the corresponding p , then, according as the elements p_k occur in $G(s)$, the group takes different forms. To illustrate in a simple case, for three colors 1, 2, 3 the group $G(2)$ takes forms

$$p_1p_1, p_1p_2, p_1p_3, p_2p_1, p_2p_2, p_2p_3, p_3p_1, p_3p_2, p_3p_3.$$

It is a general analytic representation of the frequency distribution of such forms in a set of drawings that is the first concern of the present work. The simplest case is to determine the frequency with which a form of the group $G(s)$ occurs in each $Z(n)$ where a number of draw series $Z(n)$ have been obtained by drawing say n times. In this case, there is obtained an observed distribution indicated by $U(x)$, where $U(x)$ expresses the relative frequency with which the argument x occurs, and x refers to any group form. Next, the inquiry is for that theoretical distribution $U_1(x)$ of which the observed distribution may be regarded as a sample, or for the distribution that would be obtained if the draw series $Z(n)$ were taken an infinite number of times. The form of the theoretical distribution is given by means of the ϑ -operation introduced in Bruns's treatise.

The problem is next extended in various directions. Instead of confining the inquiry to the distribution of a single group form, several forms are considered simultaneously and different weights are given to the forms considered.

These considerations suggest building an argument

$$x = a_1x_1 + a_2x_2 + \dots,$$

where a_k is the weight and x_k the frequency of the form of that weight in the totality.

It is shown that by transformations the consideration of one form for purposes of distribution may be replaced by that of one or more other forms, and that it is possible to reduce all the forms until they possess the same number of members. On account of such transformation it is sufficient to keep s fixed in $G(s)$ in treating the problem of distribution.

While the notation is rather complicated, the analytic expression for the frequency of the forms in draw groups seems to be a result fundamental in the theory of "collective quantity" (Kollektivgegenstand) in general, and for problems of statistics in particular, as drawings z_k are representative of any events back of which lies that mode of origination that belongs to problems of chance. The work appears to the reviewer to be of considerable importance for the mathematics of statistics.

H. L. RIETZ.

Elementary Treatise on the Differential Calculus. By W. W. JOHNSON. New York, John Wiley and Sons, 1908. x + 191 pp.

If phrases current in the present political situation be allowed in reviewing a text in the calculus, the best possible way to describe the impressions made on the reviewer by the present volume would be to say that it is very plainly written from the viewpoint of the "stand-patter" who refuses to be convinced of the value for purposes of instruction in the calculus of the methods of limits and function theory as promulgated by the "progressives," or of the "insurgent," methods of modern disciples of the Perry movement. And the analogy goes further than the stand-pat attitude taken on the method of rates; for it applies throughout to the contents of the 7 chapters of the volume of 191 pages.

The new text is in great part an abridgment of the author's larger treatise on the differential calculus. The contents are very similar to the old, but seemingly compounded in a more digestible form for beginners. The attitude on rates having been taken, the author naturally makes a maximum use of the student's geometric intuition in explaining the fundamental notions of the differential calculus, a point of view sometimes lost sight of by those who, regardless, hold fast to rigor of demonstration.

The derivative, or differential coefficient, is defined as the relative rate of increase of the function as compared with the

independent variable; the absolute rate is obtained when t is taken as the independent variable; and finite values having the rates for their ratio are next assigned to dx and dy ; I quote here in part from the preface.

The following chapter headings with the number of pages given to each show the emphasis placed on the several subjects: Functions, derivatives, and differentials, 43; Successive derivatives, 15; Maxima and minima, 17; Evaluation of indeterminate forms, 18; Development of functions in series, 29; Application to plane curves, 52; Functions of two or more variables, 14. From this it will be plainly seen that the evaluation of indeterminate forms and the applications to plane curves certainly receive more attention than is warranted; especially in view of the fact that indeterminate forms to be evaluated by calculus methods arise but rarely in practice and are more or less "cooked up" to suit the occasion. It is doubtful, too, if an elementary calculus is the proper place for a detailed study of the derivation of the equations and properties of the exhaustive list of higher plane curves here studied, even though these have become household words among geometers.

In the abridgment space might well have been saved by omitting entirely the brief references to pedal curves and intrinsic equations. In its place it would be possible to treat more fully such a subject as the radius of curvature, which latter seems almost lost to view.

The problems, with answers, following the several sections are of a type suitable for an elementary text in that they do not seem to be of the kind where the principles of the calculus are lost sight of in the maze of reductions involved in arriving at the answers. If any criticism were to be made, it would surely be to the effect that the answers too often "come out easy" instead of "correct to so many decimal places," a point to be considered since actual problems in the application of the calculus naturally come out in decimals.

ERNEST W. PONZER.

Hyperbolic Functions. Smithsonian mathematical tables. By GEORGE F. BECKER and C. E. VAN ORSTRAND. Washington, Smithsonian Institution, 1909. 8vo. li + 321 pp. \$4.50.

THIS volume constitutes the fourth in the set of tables for scientific investigation published by the Smithsonian Institu-

tion, the others being meteorological, geographical, and physical. It is intended to be supplementary to the logarithmic tables in use by all computers. The contents are: Table I, five place values of $\log \sinh u$, $\log \cosh u$, $\log \tanh u$, $\log \coth u$; Table II, five place values of the four natural functions; Table III, five place values of $\sin u$, $\cos u$, $\log \sin u$, $\log \cos u$, the argument u being in radians; Table IV, $\log_{10} e^u$ to seven places; Table V, five place natural logarithms; Table VI, gudermannian of u to seven places in radians and corresponding degrees, minutes, and seconds; Table VII, anti-gudermannian to hundredths of a minute (meridional parts for a spherical globe); Table VIII, radians into degrees. The introduction contains definitions and formulas.

The book is clearly printed and easy of reference. Some of the tables are borrowed, but a large number of entries are new computations either for this collection or for checking old values. The tables cannot fail to be of great service in advancing the use of the hyperbolic functions.

JAMES BYRNIE SHAW.

Problèmes et Exercices de Mathématiques générales. By E. FABRY. Paris: Hermann et Fils, 1910. 8vo. 420 pp. 10 fr.

THE volume on General Mathematics preceding this was reviewed in the *BULLETIN*, volume 15 (1909), pages 395-399. The present book contains problems and their solutions, to accompany the text proper. The problems occupy 80 pages, the solutions 336. There are 235 problems in algebra, 231 in analytics, 173 in analysis, and 90 in mechanics. These are solved in full in the remaining pages. As a whole the problems would demand considerable ingenuity on the part of the student. For an average student, who knew only what is to be found in the *Traité*, there are many which would be beyond his range. They are not in this sense exercises. But as a collection of solved illustrations of the subjects treated in the *Traité*, the book is quite valuable. The student would undoubtedly acquire considerable skill in following these models. This is an excellent storehouse from which extra problems might be drawn for the ambitious students in our American classes.

JAMES BYRNIE SHAW.

Internaciona matematikal Lexiko en Ido, Germana, Angla, Franca, e Italiana. By DR. LOUIS COUTURAT. Jena, Gustav Fischer, 1910. 4to. ii + 36 pp. M. 1.50.

WHETHER a universal language will ever be possible, and whether it will be a living tongue or an artificial construction, remains for the future to show. The fate of Volapük is still fresh in memory, and now the simple language of Esperanto is simplified still more in the new Ido. One may reflect that "le mieux est l'ennemi du bien"; what one considers simple another may not, and as soon as we begin to perfect, we begin to evolve a new thing. To be artificially universal a language must be fixed, crystallized. When one considers the inconsistent notations of mathematics itself and yet their persistence, he may take warning as to the attempt at faultless simplicity.

However, we have here a pamphlet of some thirty-six pages with five columns—Ido—Deutsch—English—Français—Italiano. The first is alphabetic and contains about 1300 words in the artificial tongue Ido. The other columns contain their equivalents. The list follows Müller's Vocabulaire closely, omitting archaic terms and those employed by a single author. The resemblance of the entire list to their French equivalents is rather striking, and one may ask whether after all, it would not be the simplest thing to adopt French as a universal tongue.

JAMES BYRNIE SHAW.

Anfangsgründe der Maxwell'schen Theorie verknüpft mit der Elektronentheorie. Von FRANZ RICHARZ. Leipzig, B. G. Teubner, 1909. ix + 245 pp.

THE elements of the theory of the electromagnetic field are now so widely taught, and to so many pupils, that numerous teachers can indulge in the satisfaction of having their own text from which their pupils may get the viewpoint which seems most desirable for them. The literature of the subject therefore grows apace, and as it grows, there devolves upon each successive author the duty to write with some well defined aim rather than to attempt an exhaustive treatment. Apparently Richarz has clearly recognized this duty and privilege and has been successful in their execution. His book must therefore be welcome and advantageous to many teachers who wish to lay the theoretical foundations for the more important applications without being led astray into discussions which at

present are chiefly of but theoretic interest. The author uses the concept of vectors freely, but hardly any vector analysis. Contrary to the growing and now usual custom of employing a right-handed co-ordinate system for electromagnetism, he uses the left-handed system of the pure mathematician and astronomer.

In Chapter I the start is made directly from the concept of a field and the Maxwell equations are obtained from the Ampère-Faraday experimental laws. The second chapter reviews and expounds and extends the analysis from the point of view of ether and matter. The next chapter treats non-conductors and static phenomena, derives Coulomb's law, discusses free and true electricity and magnetism, all with admirable clearness and precision. Chapter IV brings forward the matter of conductors and conduction currents. Ohm's law and the loss of energy from ether to matter are taken up, but without the introduction of the Poynting vector, which would introduce greater complications than are desired. It should be especially remarked that the concept of a current as a flow of electrons is clearly presented, a thing not often ventured at this stage. A following chapter introduces the potential for conductors and insulators and discusses static or stationary phenomena.

To this point about half the book has been covered. Three long chapters on electromagnetism, induction, and high frequencies complete the work. Proceeding from the general equations of the electromagnetic field, the author introduces the vector potential for a linear conductor and from it he derives the Biot-Savart law for the action between current elements. Stokes's theorem, solid angle as a potential, the multiple value of the potential, and the question of work are mentioned. The mutual potential energy of two circuits and the forces between the circuits are treated with some detail. In much the same way, the work is carried on to induction, two notable features being the derivation of induction in ponderable matter from the point of view of electron theory and the exhibition of the mass of the electron as a phenomenon of self induction. From the fact that such matters are mentioned it appears clear how careful the author is to keep before his readers the latest physical meaning attached to the fundamental phenomena of electromagnetism. High frequency phenomena in dielectrics or conductors are connected with the velocity of light, the opacity of conductors, the coefficient of reflexion, and similar questions in electromagnetic optics.

Such a mere catalog of the important subjects touched upon

by the author may seem hardly a review ; it does, however, when taken with the introductory remarks, give perhaps in the shortest space the best idea of a book of this nature where scope and presentation are the two features of most vital interest. The author certainly has written a good introduction to Maxwell's theory and has given appropriate attention to its relation to the later atomic view of electricity.

E. B. WILSON.

NOTES.

THE twenty-ninth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY will be held at the University of Chicago on Friday and Saturday, April 28-29. Titles and abstracts of papers to be presented at this meeting should be in the hands of the Secretary three weeks in advance of the meeting.

AT the meeting of the London mathematical society held on January 12 the following papers were read : By T. C. LEWIS, "A property of the number 7" ; by H. M. MACDONALD, "A mode of representation of an electromagnetic field as due to singularities distributed over the surface" ; by W. H. YOUNG, "On the fundamental theorem in the theory of functions of a complex variable" ; by E. W. HOBSON, "On the fundamental theorem relating to the Fourier constants for given functions."

THE Paris academy of sciences announces the following prize subjects for the year 1912: The grand prize (3000 francs) for the improvement of the theory of algebraic differential equations of the second or of the third order the general integral of which is uniform. Also the problem mentioned in the BULLETIN, volume 16, page 332. This was proposed before, but no suitable memoirs were received. The Francoeur prize (1000 francs) for work useful to the propagation of pure and applied mathematics. The Poncelet prize (2000 francs) for a worthy memoir in pure mathematics. The Montyon prize (700 francs) for the improvement of machinery useful in agriculture. The Fourneyron prize (1000 francs) for the best contribution to the theory of aviation. A number of prizes in astronomy are announced, without close restriction of the subject.

PROFESSOR L. HEFFTER, of the University of Kiel, has accepted a professorship of mathematics at the University of Freiburg.

DR. F. HARTOGS, of the University of Munich, has been promoted to an associate professorship of mathematics.

PROFESSORS E. JAHNKE, of the mining academy in Berlin, and E. PAPPERITZ, of the mining academy at Freiberg, have been elected to membership in the academy of sciences of Halle.

PROFESSORS J. NEUBERG, of the University of Liège, and P. MANSION, of the University of Ghent, have each been given the rank of professor emeritus. They will retire at the close of the present academic year.

PROFESSOR E. BOREL, of the University of Paris, has been appointed sub-director of the Normal School, as successor to the late Professor J. Tannery.

PROFESSOR M. DULAC, of the University of Algiers, has been appointed professor of mathematics at the University of Lyon.

PROFESSOR E. CARRUS, of the University of Besançon, has accepted the professorship of mathematics at the University of Algiers.

DR. P. BOUTROUX, of the University of Nancy, has been appointed professor of the calculus at the University of Poitiers.

DR. M. FRÉCHET has been promoted to the professorship of rational mechanics at the University of Poitiers.

DR. A. CAHEN will give a course in the theory of numbers at the University of Paris during the second semester of the present academic year.

DR. E. ZORETTI, of the University of Grenoble, will give a course in rational mechanics at the University of Caen.

DR. M. CHAZY has been appointed master of mathematical conferences at the University of Grenoble.

DR. A. CLAEYS, of the University of Ghent, has been promoted to a professorship of mathematics.

DR. M. LAGRULA will assist Professor CARTAN in the newly inaugurated course in general mathematics at the University.

of Paris. About 100 students are participating in the exercises, which consist in the evaluation of definite integrals and the use of logarithmic and other numerical tables and of calculating machines.

DR. M. MANDERLI has been appointed docent in mathematics and astronomy at the University of Bern.

AT Princeton University, Professor W. GILLESPIE has been promoted to a full professorship of mathematics.

MR. DUNHAM JACKSON, now studying at Göttingen on a travelling fellowship from Harvard, has been appointed instructor in mathematics at Harvard University for the academic year 1911-1912.

PROFESSORS E. R. HEDRICK, of the University of Missouri, and J. I. HUTCHINSON, of Cornell University, have been granted leave of absence during the second half of the present academic year; they will spend most of their time in Paris. During Professor Hedrick's absence, Professor O. D. KELLOGG will be in charge of the department of mathematics at the University of Missouri.

PROFESSORS E. J. WILCZYNSKI, of the University of Chicago, and C. L. BOUTON and J. L. LOVE, of Harvard University, are spending the present half-year abroad on leaves of absence.

THE marriage is announced of Professor E. D. ROE, JR., of Syracuse University, and Miss JOSEPHINE A. ROBINSON, professor of mathematics at Berea College, at Berea, Ky., February 1.

PROFESSOR G. GUNDELFINGER, of the technical school at Darmstadt, died December 13, 1910, at the age of 65 years.

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ERNST (P.). I: Die Aour'sche Resultantenkurve. II: Die Axoidale zweier ebener Kurven. (Progr.) Wien, 1909. 8vo. 18 pp.

- FEYER (W.). Ueber die Höldersche Funktion und einige verwandte Transcendente. Chemnitz, 1910. 4to. 41 pp. M. 2.00
- GIROD, (L.). Das sphärische Analogon der Hypocycloidenbewegung des Cardanus und sein Zusammenhang mit der Theorie eines verallgemeinerten Hooke'schen Gelenkes. (Diss.) Strassburg i. E., 1910.
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GROUPS GENERATED BY TWO OPERATORS
SATISFYING TWO CONDITIONS.

BY PROFESSOR G. A. MILLER.

(Read before the American Mathematical Society, October 29, 1910.)

THE largest group G generated by two operators s_1, s_2 satisfying two conditions of the form

$$s_1^{\alpha_1} s_2^{\alpha_2} \dots = 1, \quad s_1^{\beta_1} s_2^{\beta_2} \dots = 1$$

is, in general, of infinite order. For some special values of the exponents $\alpha_1, \alpha_2, \dots; \beta_1, \beta_2, \dots$ the order of G is necessarily finite,* but the general necessary and sufficient condition that the order of G be finite does not seem to be known. We proceed to a consideration of all the possible cases where a series of consecutive exponents is composed of numbers which have the common value unity while all of the other exponents are equal to zero, as these cases appear fundamental in the general problem.

§1. *Consecutive Exponents Equal to Unity While all the Others are Zero.*

It is easy to verify that all these possible cases are included in the following five general forms :

$$\begin{aligned} (s_1 s_2)^\alpha &= (s_2 s_1)^\beta = 1, & (s_1 s_2)^\alpha &= (s_1 s_2)^\beta s_1 = 1, \\ (s_1 s_2)^\alpha &= (s_2 s_1)^\beta s_2 = 1, & (s_1 s_2)^\alpha s_1 &= (s_1 s_2)^\beta s_1 = 1, \\ (s_1 s_2)^\alpha s_1 &= (s_2 s_1)^\beta s_2 = 1. \end{aligned}$$

As the first two conditions are equivalent to the single condition $(s_1 s_2)^\delta = 1$, δ being the highest common factor of α and β , the order of G is infinite in this case.† When $\alpha = \beta$ the fourth pair of conditions reduces also to a single condition and hence the order of G must also be infinite in this special case. We proceed to prove that G must be a cyclic group of finite order in each of the other possible cases.

To prove this fact we may first observe that G must be cyclic

* *American Journal of Mathematics*, vol. 31 (1909), p. 167.

† *Ibid.*

whenever s_1, s_2 satisfy any one of the given five pairs of conditions except the first, since the cyclic group generated by $s_1 s_2$ includes the operators s_1, s_2 in each of these cases. When either the second or the third pair of conditions is satisfied, the order of $s_1 s_2$ is explicitly a divisor of α and hence the order of G is a divisor of α . That the order of G is exactly α in each of these cases results directly from the fact that it is possible to find two operators in any cyclic group of order α such that they satisfy the conditions imposed upon s_1, s_2 . This result may be expressed as follows:

If two operators s_1, s_2 are such that identity can be obtained by taking them alternately as factors, both when we have an even number and also when we have an odd number of such factors, then they generate a cyclic group whose order divides one-half of this even number, and every cyclic group may be generated by two operators satisfying two such conditions.

To simplify the considerations as regards the two remaining pairs of conditions it may be well to observe the following elementary theorem: If α and β are any two numbers and if G is an arbitrary cyclic group it is always possible to find a pair of generating operators of G such that they satisfy the condition $t_1^\alpha = t_2^\beta$.* When the fourth pair of the given conditions is satisfied it results directly that the order of $s_1 s_2$ is a divisor of $|\alpha - \beta|$. When $\alpha = \beta$ the order of G is infinite, as was observed above. When $\alpha \neq \beta$ G is clearly a cyclic group of order $|\alpha - \beta|$, since it is possible to find two generators of such a cyclic group which satisfy the conditions imposed on s_1, s_2 by these conditions.

It remains to consider the last one of the five given pairs of conditions. Since s_1 and s_2 are commutative it results that these conditions may also be expressed as follows:

$$s_1^{\alpha+1} s_2^\alpha = s_1^\beta s_2^{\beta+1} = 1.$$

Hence

$$s_1^{\alpha+1} = s_2^{-\alpha}, \quad s_1^{\beta(\alpha+1)} = s_2^{-\alpha\beta} = s_2^{-(\alpha+1)(\beta+1)}.$$

From the last equation it results that $s_2^{\alpha+\beta+1} = s_1^{\alpha+\beta+1} = 1$. Hence the order of G is a divisor of $\alpha + \beta + 1$. Moreover, in any cyclic group whose order is $\alpha + \beta + 1$ it is possible to find two operators which satisfy these conditions and also generate the group. This proves that the order of G is exactly $\alpha + \beta + 1$ whenever s_1, s_2 satisfy this pair of conditions. The preceding considerations constitute a proof of the following theorem:

* *Quarterly Journal of Mathematics*, vol. 36 (1905), p. 51.

If identity can be obtained in two ways by forming the continued product with two operators s_1, s_2 taken alternately as factors, then the largest group G generated by s_1, s_2 is of finite order except when the number of factors in both products is even, or when the number of these factors is the same odd number in both products and the same operator occurs an odd number of times in each. When the total number of times that each factor appears in the two products is the same and the number of factors in each product is odd then G is a cyclic group whose order is this total number.*

§ 2. Conditions Imposed upon the Possible Groups by the Single Relation $(s_1 s_2)^\alpha = (s_2 s_1)^\beta$, α and β being Relatively Prime.

Since the operators s_1, s_2 are supposed to satisfy the equation

$$(s_1 s_2)^\alpha = (s_2 s_1)^\beta$$

where α, β are relatively prime, it results immediately from the fact that $s_1 s_2$ and $s_2 s_1$ have the same orders that each of these two operators generates the other. Hence we may assume that

$$s_1 s_2 = (s_2 s_1)^\gamma.$$

The group G generated by s_1, s_2 involves invariantly the cyclic group generated by $s_1 s_2$ since this cyclic group is generated also by $s_1^{-1} \cdot s_1 s_2 \cdot s_1 = s_2 s_1$. Hence G involves an invariant cyclic subgroup which gives rise to a cyclic quotient group.

Suppose that the order of $s_1 s_2$ is n . It is evident that n is prime to γ . Moreover, it is possible to select s_1, s_2 so that n is an arbitrary number prime to γ , for a generator of the cyclic group of order n is transformed into its γ th power by an operator s_2 whenever γ is prime to n . If we let t represent this generator and $s_1^{-1} t$ represent s_1 , we have a set of operators which satisfy the given conditions. The order of s_2 is a multiple of the exponent to which γ belongs modulo n , but it is not restricted in any other way. Hence the order of G is any multiple of n multiplied by this exponent, but it is not subject to any other restriction. This proves the following theorem:

If two operators satisfy the condition $(s_1 s_2)^\alpha = (s_2 s_1)^\beta$, α and β being relatively prime, the order of $s_1 s_2$ may have any arbitrary value n which is prime to α and β , and hence $s_1 s_2 = (s_2 s_1)^\gamma$. The order of s_2 is an arbitrary multiple of the exponent e to

* Cf. American Journal of Mathematics, vol. 31 (1909), p. 182.

which γ belongs modulo n , and hence the order of G , the group generated by s_1 and s_2 , is an arbitrary multiple of en . The group generated by $s_1 s_2$ is invariant under G and leads to a cyclic quotient group and hence all such groups are solvable.

From this theorem it results immediately that if two operators satisfy two conditions of the form

$$(s_1 s_2)^\alpha = (s_2 s_1)^\beta,$$

α, β being relatively prime, they may always generate an arbitrary cyclic group, since it is always possible to assume that the order of s_1 is arbitrary and that $s_1 = s_2^{-1}$. Hence two operators satisfying two such conditions must always generate a solvable group, but these conditions are insufficient to restrict the order of the possible groups. That is, two such conditions may always be satisfied by the two generators of any group in an infinite system of solvable groups of finite order.

§ 3. Consecutive Exponents Equal to Unity in Both Members of Two Equations.

At the end of the preceding section we considered the possible groups when the two conditions may be expressed as follows:

$$(s_1 s_2)^{\alpha_1} = (s_2 s_1)^{\beta_1}, \quad (s_1 s_2)^{\alpha_2} = (s_2 s_1)^{\beta_2},$$

where $\alpha_1, \beta_1; \alpha_2, \beta_2$ are two pairs of relatively prime numbers. When these exponents are not relatively prime these conditions are not generally sufficient to restrict the possible groups to solvable groups, and it is evident that whenever each member of the equations involves an even number of operators, the two operators being taken alternately, there can be no upper limit to the order of the possible groups, since any operator and its inverse could be used for s_1, s_2 . It remains therefore only to consider the cases where at least one member involves an odd number of such factors.

When one member involves an odd number of such factors while the other involves an even number the equation is evidently equivalent to one of the following:

$$(s_1 s_2)^{\alpha_1} s_1 = 1, \quad (s_2 s_1)^{\beta_2} s_2 = 1.$$

Hence (s_1, s_2) is the cyclic group generated by $s_1 s_2$. If the second conditional equation also reduced to one of the form considered in section 1, the matter would require no further

consideration. Hence we may assume that the second equation assumes the following form :

$$(s_1 s_2)^{\alpha_1} s_1 = (s_2 s_1)^{\beta_1} s_2.$$

Since the preceding condition implies that s_1, s_2 are commutative, this equation becomes

$$s_1^{\alpha_1 - \beta_1 + 1} = s_2^{\beta_1 - \alpha_1 + 1} \quad \text{or} \quad s_1^k = s_2^{-k}.$$

For the same reason the first equation must assume one of the following two forms

$$s_1^l = s_2^{1-l}, \quad s_1^l = s_2^{-1-l}.$$

Hence it results that

$$s_1^{kl} = s_2^{2l-kl} = s_2^{k-k-l} \quad \text{or} \quad s_1^{kl} = s_2^{2l-kl} = s_2^{-k-k-l}.$$

These conditions fix an upper limit for the order of s_2 and hence also for the order of the cyclic group generated by s_1, s_2 , unless $2l$ is equal either to k or to $-k$. That is, *two equations of the forms $(s_1 s_2)^{\alpha_1} s_1 = 1, (s_1 s_2)^{\alpha_1} s_1 = (s_2 s_1)^{\beta_1} s_2$ define a cyclic group of finite order unless $2\alpha + 1 = \alpha_1 - \beta_1$; and two equations of the forms $(s_2 s_1)^{\beta_1} s_2 = 1, (s_1 s_2)^{\alpha_1} s_1 = (s_2 s_1)^{\beta_1} s_2$ define a cyclic group of finite order unless $2\beta = \beta_1 - \alpha_1 - 1$* . When the special conditions are satisfied these operators generate a cyclic group whose order has no upper limit.

It remains to consider the possible groups when the two given conditions are of the forms

$$(s_1 s_2)^{\alpha_1} s_1 = (s_2 s_1)^{\beta_1} s_2, \quad (s_1 s_2)^{\alpha_2} s_1 = (s_2 s_1)^{\beta_2} s_2.$$

On multiplying the members of the first equation by the inverses of the members of the second, there results the equation

$$(s_1 s_2)^{\alpha_1 - \alpha_2} = (s_2 s_1)^{\beta_1 - \beta_2}.$$

If the two numbers $\alpha_1 - \alpha_2, \beta_1 - \beta_2$ are relatively prime, this equation implies that the cyclic group generated by $s_1 s_2$ is invariant under (s_1, s_2) , as was observed above. Since each of the two conditional equations implies that s_1, s_2 correspond to the same operator in the quotient group of (s_1, s_2) as regards the invariant subgroup generated by $s_1 s_2$, and as s_1 and s_2 must also correspond to inverse operators in this quotient group, it results that this quotient group is of order 2, if it is not

identity. Suppose that n is the order of $s_1 s_2$. As n is prime to both $\alpha_1 - \alpha_2$ and $\beta_1 - \beta_2$ according to our hypothesis, it results that a positive number x_0 less than n can be found so that $(\beta_1 - \beta_2)x_0 \equiv 1 \pmod{n}$. Hence $s_2 s_1 = (s_1 s_2)^{\beta_1}$, $\gamma \equiv (\alpha_1 - \alpha_2)x_0 \pmod{n}$. Since s_1^2 is $(s_1 s_2)^{\alpha_1}$ it is necessary that $\gamma^2 \equiv 1 \pmod{n}$. From this it results that n is a modulus of the following congruences:

$$(\beta_1 - \beta_2)x_0 \equiv 1, \quad (\alpha_1 - \alpha_2)^2 x_0^2 \equiv 1.$$

Hence $(\beta_1 - \beta_2)^2 \equiv (\alpha_1 - \alpha_2)^2 \pmod{n}$. The value of n must therefore be a divisor of $|(\beta_1 - \beta_2)^2 - (\alpha_1 - \alpha_2)^2|$, and as the order of (s_1, s_2) cannot exceed $2n$, this proves that the two conditions under consideration generally restrict the order of the group generated by s_1, s_2 . In fact, we have proved the theorem:

The largest group generated by two operators which satisfy the two equations

$$(s_1 s_2)^{\alpha_1} s_1 = (s_1 s_2)^{\beta_1} s_2, \quad (s_1 s_2)^{\alpha_2} s_1 = (s_2 s_1)^{\beta_2} s_2,$$

where $\alpha_1 - \alpha_2$ and $\beta_1 - \beta_2$ are two relatively prime numbers, is solvable and has an order which divides $2|(\beta_1 - \beta_2)^2 - (\alpha_1 - \alpha_2)^2|$.

When this group is abelian it must be cyclic.* As an illustration of the fact that two such equations may define a cyclic group and also a non-cyclic group of order $2|(\beta_1 - \beta_2)^2 - (\alpha_1 - \alpha_2)^2|$ we may use the following two sets of equations:

$$\begin{aligned} (s_1 s_2)^2 s_1 &= s_2 s_1 s_2, & s_1 s_2 s_1 &= (s_2 s_1)^3 s_2; \\ (s_1 s_2)^2 s_1 &= (s_2 s_1)^3 s_2, & s_1 s_2 s_1 &= s_2 s_1 s_2. \end{aligned}$$

From the first set we obtain $s_1 s_2 = (s_2 s_1)^{-2} = s_2 s_1$, since $(s_1 s_2)^3 = 1$; and hence the first equation reduces to $s_1^2 = 1$ while the second becomes $1 = s_1 s_2^3$ or $s_1 = s_2^3$. Hence we may assume that s_2 is of order 6 and that $(s_1 s_2)$ is the cyclic group of order 6, since such operators clearly satisfy each of the two conditional equations of the first set.

On the other hand, the second set of equations implies that $s_1 s_2 = (s_2 s_1)^3$, and hence s_1^2 is again unity. As s_1 transforms $s_1 s_2$ into its inverse, it results that (s_1, s_2) is the symmetric group of order 6 in this case. It is also easy to verify that if we assume that s_1, s_2 are two operators of order 2 whose product is of order 3, the given conditional equations will be satisfied. It should not be assumed that the order of $s_1 s_2$ is

* *American Journal of Mathematics*, vol. 31 (1909), p. 182.

always equal to $|(\beta_1 - \beta_2)^2 - (\alpha_1 - \alpha_2)^2|$. In fact, this number is 5 when $(s_1 s_2)^3 s_1 = (s_2 s_1)^4 s_2$ and $s_1 s_2 s_1 = s_2 s_1 s_2$ but $s_1 s_2$ in this case is identity, as may easily be verified. All that has been proved is that the order of $s_1 s_2$ must always divide $|(\beta_1 - \beta_2)^2 - (\alpha_1 - \alpha_2)^2|$ when the two numbers $\alpha_1 - \alpha_2, \beta_1 - \beta_2$ are relatively prime. By assigning different values to the exponents $\alpha_1, \alpha_2, \beta_1, \beta_2$ so that $\alpha_1 - \alpha_2, \beta_1 - \beta_2$ are relatively prime we may thus obtain defining relations for an indefinite number of solvable groups of finite order. For instance, it is easy to verify that the two equations

$$(s_1 s_2)^4 s_1 = (s_2 s_1)^3 s_2, \quad (s_1 s_2)^7 s_1 = (s_2 s_1)^5 s_2$$

define the group of order 32 which involves a cyclic subgroup of order 16 while the remaining operators transform the operators of this cyclic group into their seventh powers.

§ 4. *The Special Case where $(s_1 s_2)^{\alpha_1} s_1 = (s_2 s_1)^{\beta_1} s_2$ and $(s_1 s_2)^{\alpha_1 - 1} s_1 = (s_2 s_1)^{\beta_1 + 1} s_2$.*

In this special case it is evident that the second equation may be replaced by the simpler one $s_1 s_2 = (s_2 s_1)^{-1}$. Hence the first equation becomes

$$(s_1 s_2)^{\alpha_1} s_1 = (s_1 s_2)^{-\beta_1} s_2, \quad \text{or} \quad (s_1 s_2)^{\alpha_1 + \beta_1 + 1} = s_2^2.$$

Since $s_2 s_1$ is transformed into its inverse by s_2 and its $(\alpha_1 + \beta_1 + 1)$ -th power is commutative with s_2 it results that the order of $s_1 s_2$ is a divisor of $2(\alpha_1 + \beta_1 + 1)$. On the other hand, it is easy to see that s_1, s_2 may be so selected as to generate the dicyclic group of order $4(\alpha_1 + \beta_1 + 1)$, and hence the order of $s_1 s_2$ is exactly $2(\alpha_1 + \beta_1 + 1)$ whenever the operators s_1, s_2 are subject only to the two conditional equations expressed in the heading of this section. To prove this fact we may assume that $t = s_1 s_2$ is an operator of order $2(\alpha_1 + \beta_1 + 1)$ and that $s_1^{-1} t s_1 = t^{-1}$. In the dicyclic group of order $4(\alpha_1 + \beta_1 + 1)$ which involves t we may let s_1 be any operator of order 4 which is not generated by t , and then s_2 may be so selected that $s_1 s_2 = t$. The operators s_1, s_2 determined in this manner evidently satisfy the conditions imposed on $s_1 s_2$ by the equations expressed in the heading of the present section. Hence the theorem:

If two operators are restricted only by the two equations $(s_1 s_2)^{\alpha_1} s_1 = (s_2 s_1)^{\beta_1} s_2$, $(s_1 s_2)^{\alpha_1-1} s_1 = (s_2 s_1)^{\beta_1+1} s_2$ they generate the dicyclic group of order $4(\alpha_1 + \beta_1 + 1)$.

From the symmetry of the equations it results that the same group is generated by s_1, s_2 when they satisfy the two conditions $(s_1 s_2)^{\alpha_1} s_1 = (s_2 s_1)^{\beta_1} s_2$, $(s_1 s_2)^{\alpha_1+1} s_1 = (s_2 s_1)^{\beta_1-1} s_2$. The close contact of the present paper with the one entitled "Finite groups which may be defined by two operators satisfying two conditions" * should be noted.

FUNDAMENTAL REGIONS FOR CYCLICAL GROUPS OF LINEAR FRACTIONAL TRANSFORMATIONS ON TWO COMPLEX VARIABLES.

BY PROFESSOR J. W. YOUNG.

(Read before the Southwestern Section of the American Mathematical Society, November 26, 1910.)

THE purpose of this note is to call attention to a simple method for obtaining fundamental regions for cyclical groups of linear fractional transformations on two complex variables. The simplicity of the method is due to the fact that the determination of a fundamental region for a group of the specified kind is made to depend merely on the construction of such a region for a simply isomorphic group on a single complex variable. The method, moreover, may be readily extended to the case in which the number of variables is n , and to certain restricted types of groups which are not cyclical and not linear.

Let V be any (non-identical) linear fractional transformation on two complex variables, and let Γ be the cyclical group generated by V , i. e., consisting of all transformations V^n ($n = 0, \pm 1, \pm 2, \dots$). We interpret V as a collineation in a complex plane, and will consider separately each of the five types to which V may belong according to the configuration of its invariant points and lines.

Suppose first that V is of Type I, i. e., has three and only three invariant points forming the vertices of a triangle ABC . Every transformation of Γ then has the same configuration of invariant elements. The group Γ transforms the points on

* *American Journal of Mathematics*, vol. 31 (1909), p. 167.

each of the sides of the invariant triangle according to a group of projective transformations on a single complex variable, the latter group being isomorphic with Γ . Moreover, it is evident that this isomorphism is certainly not multiple if the transformations on the corresponding side of the triangle are not periodic. Suppose, then, that V produces a transformation which is not elliptic (i. e., whose characteristic cross ratio is not of the form $e^{2\pi i \theta}$, θ real) on at least one side of the triangle ABC . Let AB be such a side; the cyclical group on AB is then either hyperbolic (non-involutoric) or loxodromic and is simply isomorphic with Γ . It is well known that any cyclical group of hyperbolic or loxodromic transformations on a single complex variable is properly discontinuous;* a familiar form of a corresponding fundamental region is bounded by two chains about the double points.†

Let R_1 be a fundamental region for the cyclical group on AB , and let R_2 be the class of all points on all lines joining C to points of R_1 . R_2 is then a fundamental region for the group Γ if the lines AC and BC are regarded as singular. For if P is any point of the complex plane not on a side of the triangle ABC , the line PC meets AB in a point Q . Since R_1 is a fundamental region for the group on AB , there exists in Γ a transformation which transforms Q into a point Q' of R_1 . This transformation transforms P into a point P' on the line QC , i. e., into a point of R_2 . Further, if R_2 contained two distinct interior points P, P' equivalent under a transformation of Γ , the lines $PC, P'C$, which are necessarily distinct under the hypotheses made, would determine two distinct interior points of R_1 equivalent under the group on AB , which is impossible since R_1 is a fundamental region for the latter group.

* A group G of transformations on the points of a space S_n of n (complex) dimensions is said to be properly discontinuous in S_n , provided there exists in S_n a region E_n of n dimensions such that, (1) if P is any point of S_n not in E_n (with the exception possibly of the points of certain "singular" spreads S_k ($k < n$) in S_n), there exists in G a transformation which transforms P into a point of E_n ; and (2) two distinct interior points of E_n are not equivalent under any transformation of G . For a more complete discussion of this notion the reader may consult, e. g., Fubini, *Introduzione alla teoria dei gruppi discontinui e delle funzioni automorfe*, Pisa, 1908, pp. 142 ff.

† If the points of the complex line AB be interpreted in the usual way by the real points of a plane or sphere, the fundamental region in question is bounded by two circles about the double points (cf. Klein-Fricke, *Elliptische Modulfunktionen*, vol. 1, pp. 187 ff). We have used above the language of complex projective geometry, in which the totality of chains on a complex line correspond to the totality of real circles on the representing plane or sphere. Cf. a paper by the author, *The geometry of chains on a complex line*, *Annals of Mathematics*, 2d ser., vol. 11 (1909), pp. 33-48.

There remains in the consideration of Type I only the case in which the characteristic cross ratio on each of the sides of the triangle ABC is of the form $e^{i\theta}$, θ real. It is well known that a cyclical group of projectivities on a line generated by a projectivity whose cross ratio is $e^{i\theta}$ is properly discontinuous only if θ is commensurable with π . This is due to the fact that if the ratio $\theta : \pi$ is irrational, the corresponding group contains infinitesimal transformations, owing to the fact that under this hypothesis it is possible to determine two integers m, n such that $|m\theta + n \cdot 2\pi| < \epsilon$ for any positive ϵ . Now it is readily seen that if $e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}$ are the characteristic cross ratios on the sides of the triangle ABC , either $\theta_1, \theta_2, \theta_3$ are all commensurable with π , or else at least two of them are incommensurable with π . This follows directly from the theorem that the product of the three characteristic cross ratios taken in a proper way is equal to unity. If two of the θ 's, say θ_1, θ_2 , are incommensurable with π , it follows that Γ contains infinitesimal transformations. For it is then possible to determine three numbers m, n, n' such that the two relations $|m\theta_1 + n \cdot 2\pi| < \epsilon, |m\theta_2 + n' \cdot 2\pi| < \epsilon$ hold simultaneously.* The corresponding transformation V^n then clearly transforms any point P into an arbitrarily near point P' .

The only properly discontinuous groups of the type now under consideration are, therefore, those of finite order. If the cyclical group induced by such a Γ on one of the sides of the invariant triangle is of the same order as Γ , the same method as that used above in the hyperbolic or loxodromic cases may evidently be employed to construct a fundamental region for Γ . This method fails, however, if the order of the group on each side of the invariant triangle is less than the order of Γ .†

To obtain a fundamental region in this case we may proceed as follows: Let the order of the cyclical group on AB be m , and suppose that the order of Γ is greater than m . Let R_1 be a fundamental region of the group on AB , and let R'_1 be a fundamental region of the group on BC generated by V^* . Let R_2 be the class of all points in which a line joining C to a point of R_1 meets a line joining A to a point of R'_1 . The region R_2 is then a fundamental region for Γ . For let P be any point of the complex plane not on a side of the triangle ABC ,

* For an elementary proof of this fact, and its generalization to n irrational ratios, cf. Minkowski, *Diophantische Approximationen*, p. 7.

† This could happen, for example, if the orders of the groups on the sides of ABC were respectively 15, 10, and 8 (since $\frac{1}{15} + \frac{1}{10} - \frac{1}{8} = 0$).

and let the line PC meet AB in Q . There then exists in Γ a transformation V^a which transforms Q into a point Q' of R_1 . The point P is transformed by V^a into a point of the line $Q'C$. If P' is not a point of R_2 , let the line $P'A$ meet BC in S . There then exists a transformation of the cyclical group generated by V^a , i. e. a transformation V^{mb} , which transforms S into a point S' of R_1 . This transformation will transform P' in to the intersection P'' of $Q'C$ and $S'A$, since V^{mb} leaves the line $Q'C$ invariant. But P'' is by definition a point of R_2 . On the other hand, suppose R_2 contained two interior points P , P' which are equivalent under a transformation V^a of Γ . If the lines PC , $P'C$ are distinct, the hypothesis would imply the existence of two distinct interior points of R_1 equivalent under Γ , which contradicts the assumption that R_1 is a fundamental region of the group on AB . If, however, P , P' are collinear with C , the lines PA , $P'A$ are distinct. Since the transformation transforming P into P' now leaves PC invariant, it is of the form V^{am} ; but this would imply the existence in R_1 of two points equivalent under a transformation of this form, which again contradicts the assumption regarding R_1 . This completes the proof that R_2 is a fundamental region for Γ .

The points on two of the sides of the invariant triangle have in the above discussion been left out of account; i. e., we have regarded the lines in question as singular. It is clear, however, that if the groups generated on such an invariant line are properly discontinuous, we need only add to R_2 a fundamental region for every such group on an invariant line in order to have a region which is fundamental for all points which are transformed in a properly discontinuous way.

The discussion of the remaining four types now offers no difficulty, as the first method described above will readily apply. This method may be described in slightly different language as follows: The transformation V transforms the lines through any invariant point according to a group isomorphic with Γ . If this isomorphism is simple, we choose a *fundamental region of lines* for the group on the invariant point. The region consisting of all the points on these lines is a fundamental region for Γ . If V is of Type II, i. e., has two and only two invariant points, the group on one of these points is parabolic, and hence (since it is not of finite order) is simply isomorphic with Γ . If V is of Type III, i. e., has one and only one invariant point, the group on this point is likewise simply isomorphic with Γ . If

V is of Type IV (a homology) the group induced by Γ on any point of the axis of homology is simply isomorphic with Γ (it may not be properly discontinuous, however, in which case Γ has no fundamental region). Finally, if V is of Type V (an elation), the induced group on any point of the axis distinct from the center is parabolic and simply isomorphic with Γ . The method then suffices to determine a fundamental region for every properly discontinuous cyclical group of linear fractional transformations on two complex variables.

It is readily seen, moreover, that the method may be extended to the case where the number of variables is n . It will serve also to simplify the problem of determining a fundamental region of any properly discontinuous group of transformations on the points of an S_n which leaves a point of S_n invariant, provided the group on the S_{n-k} (for some $k = 1, 2, \dots, n-1$) through this invariant point is simply isomorphic with the given group. The simplification consists in reducing the problem to the determination of a fundamental region for a simply isomorphic group on a smaller number (namely, k) of variables.

LAWRENCE, KAN.,
November, 1910.

ON THE RELATIVE DISCRIMINANT OF A CERTAIN KUMMER FIELD.

BY PROFESSOR JACOB WESTLUND.

(Read before the American Mathematical Society, September 7, 1910.)

In a paper published in the *Transactions of the American Mathematical Society* for October, 1910, I determined the fundamental number or discriminant of the algebraic number field $k(\sqrt[p]{m})$, generated by the real p th root of the positive integer m . Denoting this discriminant by d and setting $m = a_1 a_2^2 \dots a_{p-1}^{p-1}$, where $a_1 a_2 \dots a_{p-1}$ is not divisible by the square of a prime, the following result was obtained:

$$(1) \quad d = (-1)^{(p-1)/2} p^{p-2} (a_1 a_2 \dots a_{p-1})^{p-1}$$

or

$$(2) \quad d = (-1)^{(p-1)/2} p^p (a_1 a_2 \dots a_{p-1})^{p-1},$$

according as $b^{p-1} - a_{p-1}^{p-1}$ is divisible by p^2 or not, where $b = a_1 a_2^2 \dots a_{p-2}^{p-2}$.

By adjoining $\rho = e^{2\pi i/p}$, p being an odd prime, to this field we obtain the Kummer field $k(\sqrt[p]{m}, \rho)$. The object of the present note is to make use of the results mentioned above in determining the relative discriminant D_r of this Kummer field with respect to the subfield $k(\sqrt[p]{m})$.

Denoting by D the discriminant of $k(\sqrt[p]{m}, \rho)$, we have *

$$(3) \quad D = d^{p-1} N(D_r).$$

Similarly

$$(4) \quad D = d'^p N(D'_r),$$

where d' is the discriminant of $k(\rho)$ and D'_r the relative discriminant of $k(\sqrt[p]{m}, \rho)$ with respect to $k(\rho)$. But $d' = (-1)^{(p-1)/2} p^{p-2}$ †, hence

$$(5) \quad D = (-1)^{(p-1)/2} p^{p(p-2)} N(D'_r).$$

We have a general method‡ for determining the relative discriminant of any Kummer field with respect to $k(\rho)$, but usually the method is of very little practical value. In this particular instance, however, we are able to apply the general method. Two cases arise according as m is divisible by p or not.

I. m not Divisible by p .

Let c be any prime factor of m and $e'(i < p)$ the highest power of c contained in m . Then

$$c = P_1 P_2 \dots P_e,$$

where $P_1, P_2, \dots, P_e$ are prime ideals in $k(\rho)$ of degree f , and $p-1 = ef$. Now m contains $P_l^{e'}$ ($l = 1, 2, \dots, e$), and since i is prime to p it follows that c^{p-1} is the highest power of c contained in D'_r . Hence

$$(6) \quad D'_r = (a_1 a_2 \dots a_{p-1})^{p-1} \lambda^n,$$

where $\lambda = (1 - \rho)$.

To determine the exponent n we proceed as follows. Let s be the highest exponent $\leq p$ for which there exists a number α

* Hilbert, *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 4 (1894-95), p. 206.

† Hilbert, p. 327.

‡ Hilbert, pp. 393-394.

in $k(p)$ such that

$$m \equiv \alpha^p \pmod{\lambda^s}.$$

If $m^{p-1} \equiv 1, \pmod{p^2}$, then $s = p$. For if $\alpha \equiv m, \pmod{\lambda}$, we have

$$\alpha^p \equiv m^p, \pmod{\lambda^p}$$

and hence

$$\alpha^p \equiv m, \pmod{\lambda^p},$$

since $p = \lambda^{p-1}$. In this case it follows that D'_r is prime to p and hence $n = 0$.

If $m^{p-1} \not\equiv 1, \pmod{p^2}$, then $s = p - 1$. For if $s = p$ we should have

$$\alpha^p \equiv m, \pmod{\lambda^p},$$

and hence

$$\alpha \equiv m, \pmod{\lambda},$$

from which follows that

$$m^p \equiv m, \pmod{p^2},$$

which is contrary to the hypothesis. Hence in this case $\lambda^{2(p-1)}$ is the highest power of λ contained in D'_r . Hence $n = 2(p - 1)$.

But if $m^{p-1} \equiv 1, \pmod{p^2}$, we should have

$$b^p a_{p-1}^{p(p-1)} \equiv b a_{p-1}^{p-1}, \pmod{p^2},$$

since $m = b a_{p-1}^{p-1}$, and hence

$$b^{p-1} \equiv a_{p-1}^{p-1}, \pmod{p^2}.$$

And conversely if $b^{p-1} \equiv a_{p-1}^{p-1}, \pmod{p^2}$, it follows that $m^{p-1} \equiv 1, \pmod{p^2}$. Hence

$$(7) \quad D'_r = (a_1 a_2 \cdots a_{p-1})^{p-1}$$

if $b^{p-1} - a_{p-1}^{p-1}$ is divisible by p^2 , and

$$(8) \quad D'_r = (a_1 a_2 \cdots a_{p-1})^{p-1} \lambda^{2(p-1)}$$

if $b^{p-1} - a_{p-1}^{p-1}$ is not divisible by p^2 .

II. m Divisible by p .

If c be a prime factor of m different from p , it is evident that c^{p-1} is the highest power of c contained in D'_r . And if p^i is the highest power of p in m , it follows, since i is prime to p ,

that λ^{p^2-1} is the highest power of λ contained in D_r . Hence

$$(9) \quad D_r = c^{p^2-1} \lambda^{p^2-1},$$

where c is the product of the distinct prime factors of m different from p .

From (1), (2), (3), (5), (7), (8), and (9) it then easily follows in both cases that

$$(10) \quad N(D_r) = p^{p-2}.$$

But in the field $k(\sqrt[p]{m})$ we have the following decomposition of p into prime ideal factors, as was proved in the paper mentioned above:

$$(11) \quad p = P^p,$$

if $b^{p-1} - a_{p-1}^{p-1}$ is not divisible by p^2 , and

$$(12) \quad p = P^{p-1} Q,$$

if $b^{p-1} - a_{p-1}^{p-1}$ is divisible by p^2 , where P and Q are different prime ideals of the first degree.

In the first case we obtain

$$(13) \quad D_r = P^{p-2}.$$

In the second case, however, our method does not enable us to determine the exact powers of P and Q which enter into D_r . We only know that

$$(14) \quad D_r = P^x \cdot Q^y,$$

where $x + y = p - 2$.

PURDUE UNIVERSITY,
October, 1910.

NOTE ON RECIPROCAL FIGURES IN SPACE.

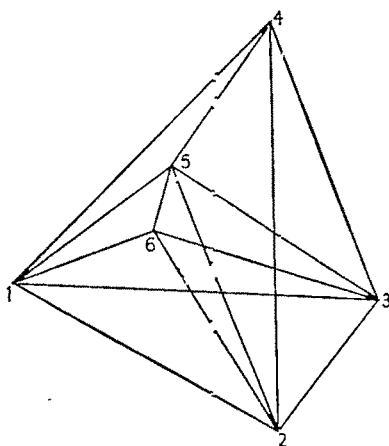
BY PROFESSOR PETER FIELD.

MAXWELL [Collected Works, page 523. Also see Rankine, *Philosophical Magazine* for February, 1864] defines figures in three dimensions as reciprocal when they can be so placed that every line in the one is perpendicular to a plane face of the other and every vertex in the one is represented by a closed polyhedron with plane faces in the other.

The simplest case [Maxwell, loc. cit., page 524] is that of 5

points in space with their 10 connecting lines. This forms a figure composed of 5 tetrahedrons and having 10 triangular faces. The 5 points which are the centers of the spheres circumscribing the 5 tetrahedrons, when connected, form another figure which is reciprocal to the first, and stresses proportional to the areas of the faces of one figure when applied along the edges of the other will keep its vertices in equilibrium. In either one of these figures, 4 of the points might be considered as the vertices of a tetrahedral frame and the fifth point as the point of concurrence of 4 forces which keep the frame in equilibrium. The reciprocal figure then exhibits the magnitudes of the forces and the stresses in the different members of the frame.

Although the idea of reciprocal figures in space is not of much importance in mechanics, because a reciprocal figure can be constructed only in case certain conditions are satisfied, nevertheless it seems of some interest to construct a figure which can be interpreted as representing a frame held in equilibrium by four nonconcurrent forces and for which a reciprocal figure can be constructed.



equilibrium by four nonconcurrent forces and for which a reciprocal figure can be constructed.

In the figure at the left, 1, 2, 3, 4 represents a given frame. The line of action of the force at 4 is taken as 45, 5 being any point on the line. As the forces acting at the 4 vertices are in equilibrium, they lie on a hyperboloid [Föppl, Technische Mechanik, volume II, page 158] and it is therefore possible to draw a line through 5 which meets the lines of action of the three remaining forces.

Let 6 be any point on this line and join the points as in the figure. The stresses in 15 and 16 can be chosen arbitrarily (as the whole figure regarded as a frame has two more edges than are necessary to make it rigid) and the remaining stresses are then fixed.

The figure can be considered as made up of the tetrahedrons

1234, 1236, 1256, 1356, 2356, 1254, 1354, and 2354. About each of these tetrahedrons a sphere can be circumscribed and the centers of these spheres could be taken as the vertices of a figure which is reciprocal to the given one, i. e., it would be a possible solution. The most general reciprocal figure which can be constructed must contain arbitrary constants corresponding to the fact that the stresses along 15 and 16 can be chosen arbitrarily. This is brought out clearly by constructing the reciprocal figure by the method used by Klein and Wieghardt (*Archiv der Mathematik und Physik*, 3 Reihe, Band 8, Heft 2) which may be summarized as follows:

There are given two sets of rectangular axes that are parallel. Assume a system of cells in the x, y, z space which are so arranged that they fill a certain boundary polyhedron doubly. Assign to each cell C_j a linear function

$$t + \theta_j = \xi_j x + \eta_j y + \zeta_j z.$$

To the cell C_j in the x, y, z space there corresponds a point $P_j \equiv (\xi_j, \eta_j, \zeta_j, \theta_j)$ whose projection in the ξ, η, ζ space has the coordinates ξ_j, η_j, ζ_j . Similarly, to the cell C_k there corresponds the point P_k . If C_k and C_j are neighboring cells, the plane between the two has the equation

$$(\xi_j - \xi_k)x + (\eta_j - \eta_k)y + (\zeta_j - \zeta_k)z - (\theta_j - \theta_k) = 0,$$

which is a plane perpendicular to the line $P_j P_k$. If the origin is taken as a vertex in the reciprocal diagram, $t = 0$ for the corresponding cell. From the equations of the boundary planes it is possible to find the values of t (and consequently the coordinates of the vertices in the reciprocal diagram) for the other cells. In this way the reciprocal figure can be constructed.

In order to prevent the detail work from becoming too cumbersome, take the frame as $P_1 \equiv (0, 0, 0)$, $P_2 \equiv (1, 0, 0)$, $P_3 \equiv (0, 1, 0)$, $P_4 \equiv (0, 0, 1)$, $P_5 \equiv (1, 1, 1)$, and $P_6 \equiv (1, -2, 1)$. The relation between the figure and its reciprocal can then be tabulated as follows:

Cell	Value of t	ξ	η	ζ	θ
1234	0	0	0	0	0
5123	bx	0	0	b	0
5134	bx	b	0	0	0
5124	by	0	b	0	0
6235	$c(x+y) + (b-c)z - c$	c	c	$b-c$	c
6345	$(b-c)x + c(y+z-1)$	$b-c$	c	c	c
6234	$(x+y+z-1)(3c-b)$	$3c-b$	$3c-b$	$3c-b$	$3c-b$
6245	$(z-c)(x+z-1) + cy$	$b-c$	c	$b-c$	$b-c$

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MATHEMATICAL PHYSICS FOR ENGINEERS.

Einführung in die Theorie des Magnetismus. Von R. GANS.
Leipzig, Teubner, 1908. vi + 110 pp.

Einführung in die Maxwell'sche Theorie der Elektrizität und des Magnetismus. Von CL. SCHAEFER. Leipzig, Teubner, 1908. viii + 174 pp.

Funktionentafeln mit Formeln und Kurven. Von E. JAHNKE und F. EMDE. Leipzig, Teubner, 1909. xii + 176 pp.

PROFESSOR JAHNKE is editing a series of texts for engineers. The exact title of the series is Mathematisch-physikalische Schriften für Ingenieure und Studierende. It is published by the firm of B. G. Teubner with the usual typographical excellence of all such works issued by this house. The titles of the volumes which have already appeared, in addition to the three at the head of this review, are Elektromagnetische Ausgleichsvorgänge in Freileitungen und Kabeln, Die Theorie der Besselschen Funktionen,* Die Vektoranalysis und ihre Anwendungen in der theoretischen Physik (2 volumes),† Theorie der Kräftepläne.‡ There are further announced some twenty-five volumes to which titles and authors have already been assigned, and we are assured that a longer continuation of the series is in contemplation. The texts, or should they be called tracts, are all of moderate size, varying from a bit over 100 to somewhat less than 200 pages. From those which have hitherto appeared it is already evident that each has a definite aim and mission to fulfil, which indeed it does fulfil, and that the series as a whole, if not carried so far as to lose its vitality, will form a

* Reviewed in this BULLETIN, volume 16 (1910), p. 385.

† Reviewed in this BULLETIN, volume 17 (1910), p. 100.

‡ Still others have appeared since the date of this review.

valuable collection for students, engineers, and (let us add) mathematicians.

As it is chiefly mathematicians to whom this review is addressed, it will not be amiss to state why this series of tracts seems to us particularly useful to them. In the first place mathematicians, as a general rule in this country at least, are certainly not over-familiar with mathematical physics. The doctorate is given at our universities to many a student of mathematics who has hardly a respectable knowledge of mechanics — to say nothing of the mathematical theories of optics, electromagnetism, thermodynamics, and other physical sciences. The specialist thus turned out may be very highly trained, indeed somewhat fine from over-training, but he is essentially uneducated, essentially narrow. He is unable to appreciate where a great deal of his mathematics arose and what were the physical problems which necessarily called it into being. Moreover, he is really not infrequently as yet unfit to teach with a broad unprejudiced view the elementary calculus, elementary mechanics, and advanced calculus which he may be called upon to teach. Under these circumstances it is but small wonder that many leaders in the various departments of our scientific and engineering schools take an indifferent if not antagonistic attitude toward mathematical instruction as it is. In the second place it is very difficult for the young mathematician who feels the inadequacy of his training upon the practical side to make up his deficiency. The massive works on mathematical physics are written for those accustomed to such subjects and are as difficult and perplexing reading to the mathematician as the modern mathematics is to the physicist. What is needed is a series of tracts, written in a simple explanatory style and each dealing with some well connected and restricted field. Toward mathematical physics the mathematician is really in the position of a *Studierender*. To us Jahnke's series seems admirably fitted not only to serve the class for whom it is intended but to render great service in broadening such of our young pure mathematicians as would desire to extend their knowledge in the direction which is most likely to add life and value to their teaching and sympathy toward their colleagues in technical departments.

In proceeding to the review of the three particular volumes which are now to be considered we shall adopt the chiasmic order

and begin with the last. Jahnke and Emde's table of functions and graphs is a book which is likely to find an active working place in the library of any one who has often to reduce problems to numerical answers. The chief tables are essentially four-place tables. Four-figure accuracy is generally sufficient and should be adopted as the standard of the engineer, mathematician, and physicist. There are some problems which require a closer solution than one which allows an error of one or two hundredths of one per cent; but these are few and should not be put in the foreground. A volume of five-place tables of the functions which Jahnke and Emde abulate with graphs in 176 pages would be of unwieldy size and serve few purposes for which the present work is not available. The type here selected is large, heavy, and easily read; but a paper not quite so shining would have materially lessened the eye-strain attendant upon long continued use of the table. The cuts and rulings upon which they appear are clear and can easily be read graphically for rough work. Moreover, even when the cuts are not to be used for graphical work, they are very useful in showing the general shape of the function tabulated; in some instances, however, we believe that the graphs could just as well have been omitted.

The first table is of $x \tan x$ and $x^{-1} \tan x$ and is very brief; the values of x are at each tenth (in radians) from 0.0 to 1.5. The authors remark below the graphs that: "In order to obtain intermediate values of these functions it is best to plot off the numbers here given on millimeter paper and to read the desired values from the curves thus obtained. A similar remark applies to all later tables in which the interval is taken too large." We do not understand this; it would seem as though if the interval is too large, a replot from the tables would be far worse than reading the curve as it is; what really is needed is a replot with a large number of intermediate values which are not here given in the table. In fact this particular table seems useless except for very rough work; arithmetic interpolation either forward or backward is of course out of the question. Table II contains the roots of some transcendental equations,

$$\tan x = x, \quad \tan x = \frac{2x}{2 - x^2}, \quad \cos x \cosh x = \pm 1,$$

and others. Not only are the roots tabulated, but the formulas from which they may be computed are often given. Table III

is for the interchange of $a + bi$ and re^{ψ} . The transformation

$$b/a = \tan \psi, \quad a + bi = a \sec \psi e^{i\psi}$$

is made and the values of $\sec \psi$ and of ψ are tabulated according to the values of $\tan \psi$. If b and a are small numbers so that division and multiplication are easy, this is the best way to tabulate; but if a , b are any four-figure numbers, it would probably be better to tabulate $\log \sec \psi$ and ψ . Table IV gives

$$\frac{\pi}{2}x, \quad e^x, \quad e^{-x}, \quad \frac{e^x}{\sqrt{\frac{1}{2}\pi x}}, \quad \frac{e^{-x}}{\sqrt{\frac{1}{2}\pi x}}.$$

The values of x run by alternate tenths from 0.0 to 6.0. The interval between the entries for x is so great that here again arithmetic interpolation is out of the question. It will therefore be seen that although the functions are given to four figures, the argument is given to only two and this table is not in any true sense a four-place table. A number of the tables are constructed in this manner; the authors are apparently relying on a replot of the tabulated values and a graphical interpolation from the curves thus obtained. For many purposes this is perhaps sufficient.

Table V is on the hyperbolic functions. Here there are eight pages of formulas giving the definition of the functions and the relation between the different functions, the addition theorems, the functions of multiple angles, the powers of $\sinh x$ and $\cosh x$, the relation to the exponential, logarithmic, and circular functions, the derivatives of the functions and integrals which contain them either in the integrand or in the primitive, and also approximations to the functions. The only numerical tabulation is of the gudermannian angle γ , which is to seconds while the argument advances by hundredths from 0.00 to 5.00 and by 0.05 from 5.00 to 8.00. It is therefore apparent that to obtain $\sinh x$ and $\cosh x$ it is necessary to take $\tan \gamma$ and $\sec \gamma$ out of a trigonometric table. In view of the large amount of room given to the formulas and of the convenience and importance of the functions in numerous practical problems, we believe it would have been well also to tabulate $\sinh x$ and $\cosh x$ or their logarithms directly to four figures as is done in B. O. Peirce's Short Table of Integrals. At this point let us remark that we believe all students of engineering should be

familiar with the use of tables of hyperbolic functions; in some schools these functions are a regular part of the curriculum, and in no case would it require more than a lesson or two to introduce them.

Tables VI and VII give the important functions

$$\text{Si } x = \int_0^x \frac{\sin x}{x} dx, \quad \text{Ci } x = - \int_x^\infty \frac{\cos x}{x} dx \quad (x > 0),$$

$$\text{Ei } x = \int_{-\infty}^x \frac{e^{-x}}{x} dx = \text{li } e^x \quad (x > 0 \text{ and } x < 0),$$

$$\int_0^x \cos \frac{1}{2} \pi x^2 dx = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\cos z}{\sqrt{z}} dz,$$

$$\int_0^x \sin \frac{1}{2} \pi x^2 dx = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\sin z}{\sqrt{z}} dz.$$

Not only are the functions tabulated; the direct and asymptotic series and several approximate formulas which may be used for the purpose of computing the functions are given. In like manner Table VIII, which gives the Γ -function, its logarithmic derivative and some related functions, contains a large number of the important formulas involving the Γ -function. Table IX is a very complete table of the error function and its first six derivatives. In Table X is found Pearson's function

$$\mathcal{F}(\nu, \nu) = e^{-1/\nu} \int_0^\nu \sin^\nu x e^{\nu x} dx,$$

and the logarithms of the first 31 Bernoullian numbers. Apart from their practical services, these tables, VI-X, will be found useful to the teacher for various exercises in advanced calculus when dealing with functions defined by integrals. One of the best things a student can learn is how to compute the numerical value of a given function which occurs in his work; he is not so liable to get lost in his subject nor to worry so much over the rigorous end of it.

The elliptic functions and integrals are treated in Table XI which extends to 54 pages. Numerous introductory formulas are given connecting the functions $F(k, \phi)$, $E(k, \phi)$, $\text{sn } u$, $\text{cn } u$, $\text{dn } u$, ϑ , ϑ_1 , ϑ_2 , ϑ_3 , p , ξ , σ . The elliptic integrals $F(k, \phi)$ and $E(k, \phi)$ are tabulated at intervals of 5° for $\alpha = \sin^{-1} k$ and of 1° for ϕ .

The complete integrals are tabulated separately and, as α nears 90° , the interval is gradually cut down to $6'$. For computation with the aid of the ϑ -functions and especially for all formulas involving q , the values of $\log q^{-1}$ are given at intervals of $5'$ for α . Brief tables of the ϑ -functions are also constructed. As for the Weierstrassian functions, $\wp$, $\wp'$, ζ , σ are tabulated in the equianharmonic case

$$g_2 = 0, \quad g_3 = 1, \quad u = \frac{r\omega_2}{180}, \quad \omega_2 = 1.53995$$

for every unit from $r = 0$ to $r = 240$. Tables for the mutual induction and attraction of two coaxial circular currents complete the set. The only comment we would make is to commend the table giving $\log q^{-1}$. Despite the trigonometrical analogies of the functions sn , cn , dn and despite the beauties of $\wp$, ζ , σ , we believe that the functions which should be emphasized in discussing elliptic functions and integrals are the ϑ -functions. In most practical problems the convergence of the series involving q is so rapid that only a very few terms are required. Moreover, for theoretical purposes it is well to familiarize the student early in his work with integral transcendental functions, which play such an important rôle in analysis and of which the ϑ -functions are among the simplest and most useful examples.

After Table XII on zonal harmonics, the volume closes with Table XIII on the Bessel functions. It is a long drawn-out close of 85 pages — nearly half the volume. About all the formulas that one could want for the practical use of the Bessel functions, including a long array of differential equations solvable in terms of them, are tabulated. The values of $J_n(x)$ for $n = \pm \frac{1}{2}$ and $\pm \frac{1}{2}, \pm \frac{3}{2}, \dots, \pm \frac{13}{2}$ are given; for $n = \pm \frac{1}{2}$ the interval in x is 0.2, for the others it is 1.0 from 0 to 50. Just what is the value of a table of $J_{n+\frac{1}{2}}(x)$ to four significant figures when the interval in x is so great as to give only six points on each complete oscillation of the curve? And are we supposed to plot off these values, make a graph, and interpolate graphically? We do not know. Perhaps it is only for integral values of x that $J_{n+\frac{1}{2}}(x)$ is needed. Tables of $J_0(x)$ and $-J_1(x)$ at intervals of 0.01 from 0.00 to 15.50 are offered; these are real four-place tables. Values of $Y_0(x)$, $-Y_1(x)$, $K_0(x)$, $K_1(x)$, $N_0(x)$, $-N_1(x)$, $J_0(ix)$, $-iJ_1(ix)$, $\frac{1}{2}i\pi H_0^{(1)}(ix)$, $-\frac{1}{2}i\pi H_1^{(1)}(ix)$, $J_0(x\sqrt{i})$, $\sqrt{x}J_1(x\sqrt{i})$, $N_0(x\sqrt{i})$, $H_0^{(1)}(x\sqrt{i})$, $H_1^{(1)}(x\sqrt{i})$ are set down at less frequent intervals and over

a more restricted total range; also ber x , bei x , and tables for the self-induction and resistance in a straight circular wire carrying an alternating current. Numerous sets of roots of $J_n(x) = 0$ are also to be found. We note further some tables of $J_n(x)$ for integral values of n from 1 to 24 and for such integral values of x as make $J_n(x) > 10^{-18}$; and there are still other tables. It will thus be seen that the subject of Bessel functions is very thoroughly covered, perhaps disproportionately so. We should remember, however, that the Bessel functions are very important in many applications of mathematics; apparently they are more important than the elliptic functions. Indeed if one had to choose between some mention of the Bessel functions and equal mention of elliptic functions in a second course in calculus, one should unhesitatingly choose the former, especially if the time were somewhat restricted. It has always seemed somewhat peculiar and unfortunate that B. O. Peirce in extending his Short Table of Integrals in its excellent revised edition should have added some 85 formulas on elliptic functions while having the Bessel function represented only by the definition of $J_n(x)$ when n is integral.

From these extended comments it should appear that we have in Jahnke and Emde's *Funktionentafeln* an extremely useful collection of formulas and values relating to the most important functions other than elementary, that the chief tables are suited for four-place accuracy, and that there are numerous other tables sufficiently good for graphical or slide-rule work. There can be little doubt that the use of the book will be very general among all workers with mathematics.

To review Schaefer's tract on the introduction to Maxwell's theory without appearing so complete and enthusiastic an admirer of the work as to lessen the value of the review is an impossibility; the work seems to have nothing but excellencies. In fact as soon as it came to our notice we immediately introduced it to our classes and followed it closely from start to finish. We shall merely try here to explain the reason for our enthusiasm. Every one is probably familiar with at least the covers of a number of works on Maxwell's theory; those who have sought familiarity with the interior know full well the difficulties encountered; the works err either toward giving a very gross physical and equally restricted mathematical explanation

of the actions in the ether or toward amassing great arrays of mathematical formulas with too little attention to the physics involved. So far as we know, Maxwell in his original treatise strikes a much happier mean between the extremes than his numerous followers and 'expounders.' Schaefer's text is a sort of minor Maxwell, an exposition eminently suitable for students and engineers — and mathematicians. Like Maxwell, Schaefer appreciates the great advantages derivable from using a vector analysis; but like him, he does not wish to assume the necessary knowledge on the part of his readers. Further, the author decides not to use Green's and Stokes's theorems on divergence and curl, but to develop their equivalents directly on the fields he is considering whenever he finds it necessary to obtain results which the theorems would immediately yield. This course of procedure will undoubtedly be the more satisfactory to the greater number of readers of the work.

Chapter I is on electrostatics. After a few words in the way of describing fundamental facts and giving necessary definitions, the author mentions the fluid hypotheses and Coulomb's law for point charges. He then starts in upon the question of action at a distance versus action in a medium. He shows that the field $\mathbf{E}$ which arises from any distribution of charged points satisfies the equation $\text{div } \mathbf{E} = 0$ except at the points; this he calls the first law of action in the medium. By introducing Gauss's theorem that the induction of $\mathbf{E}$ through a surface is 4π times the total charge within, the further result $\text{div } \mathbf{E} = 4\pi\rho$ is obtained. He next shows that the field due to any set of point charges has a potential so that $\mathbf{E} = -\nabla\phi$ or $\nabla \times \mathbf{E} = 0$ or the line integral of $\mathbf{E}$ about a circuit is zero. This he calls the second law of action in the medium and he derives the relation $\nabla \cdot \nabla\phi = -4\pi\rho$. The peculiar beauty of this development is the clear and artful way in which mathematics founded on the conception of point charges is combined with physics founded on the conception of action in a medium. The idea of a medium is given further emphasis by the construction of a hydrodynamic analogy. After a few applications to condensers the author takes up homogeneous and non-homogeneous dielectrics, discusses the difference between free and true electricity, and gives additional applications to condensers. A short section on the energy of the field closes the chapter; but the section is not so short that the author cannot show how in the simplest case the field theory of energy leads to the same value as the

theory of action at a distance. That he cannot handle the general theorem is due merely to the unavailability of Green's theorem. The reader of this first chapter ought easily to obtain the idea that action at a distance and action through a medium are two different points of view in accounting for the same phenomena, that they lead to the same mathematical formulation of the problem, that through their mathematics they are interchangeable, and that, as they are thus far equivalent, the reader may feel free to adopt as definitive either hypothesis that future investigations may establish as preferable.

A brief chapter on magnetostatics constructed on the model of the previous one leads to the discussion of the electric current and its magnetic field in Chapter III. The same general method of development is also used here. The field $\mathbf{H}$ due to a straight current is shown to fulfil the field equation $\nabla \cdot \mathbf{H} = 0$, or better $\nabla \cdot \mathbf{B} = 0$ when μ varies. The law that the circuit integral of $\mathbf{H}$ is proportional to the current through the circuit is discussed for the same simple case. It is then stated that these laws hold for any field. What amounts nearly to a demonstration of Stokes's theorem is then given to establish the equations $4\pi c^{-1}\mathbf{C} = \nabla \times \mathbf{H}$. The equation $\nabla \cdot \mathbf{C}$ which follows as a consequence is explained. The vector potential is introduced and the Biot-Savart law for the mutual action of current elements is deduced. Then follow the Maxwell equations $c\mathbf{E} + 4\pi\mathbf{C} = c\nabla \times \mathbf{H}$, the discussion of Ohm's and Joule's laws, and a careful treatment of the electrostatic and electromagnetic systems of units. It may be observed that Schaefer uses the mixed system. Chapter IV on induction carries on the work to find the equations $\dot{\mathbf{B}} = -c\nabla \times \mathbf{E}$, to integrate the equations by potentials, to present Poynting's theorem and explain its significance, to deduce the self and mutual induction of circuits, and to treat in considerable detail the phenomena connected with electric oscillations in simple and coupled circuits. Chapter V on electric waves deals with plane waves in isotropic dielectrics, reflection and refraction, (for which the laws are obtained by using the previously established boundary conditions for continuity or discontinuity of the components of $\mathbf{D}$, $\mathbf{E}$, $\mathbf{H}$, $\mathbf{B}$,) and finally the question of electromagnetic waves in metals. There is a great mass of vital electromagnetic physics in this chapter and it is all presented with the author's usual clearness and care in exposition. The reader can hardly get the wrong or fail to get the right point of view.

If this brief summary of the contents of Schaefer's book can serve to justify so unreserved enthusiasm for the work, we shall be happy ; we shall be happier if it can induce others to read the book with interest and study it with care. There can now be no sufficient excuse for lack of familiarity with the elements of Maxwell's great theory ; this presentation is too simple, too perfect to offer any difficulties ; it neither offers too much nor affords too little for a proper introduction to the subject. Jahnke is to be felicitated on securing such a tract for his series ; we only wish it had appeared in English.

Gans's tract on magnetism is also written from the point of view of Maxwell's theory, which as is stated in the preface is the point of view more and more adopted in technical work. There is consequently some duplication between this and the previous volume reviewed. Another point of the preface which is noteworthy is that Gans like Schaefer would be very glad to use vector analysis if he quite dared. The cry for some sort of vector analysis is growing as the waste of space and energy and the danger of an incorrect and insufficiently physical view, which arise when cartesian analysis is used, become more prominent. It is probable that mathematicians at large would do well to familiarize themselves with the elements of vector analysis for the purpose of seeing how the little that is needed so badly can best be worked into the courses in mathematics. The plan of giving vectors as a separate and somewhat elaborate course in a few institutions or even in many institutions is not a success ; it does not meet the need. Not long ago one of our colleagues, not himself a mathematician, expressed the conviction that all of our engineering students who carry their mathematics into the third year should probably have some vector analysis and that the electrical engineers should certainly have it. In this we entirely concur. And we surely believe that the vectors should come in little by little throughout the student's course in mathematics ; the time could readily be saved from analytic geometry, especially from solid analytic geometry. The mere notion and notation of the scalar product of two vectors will of itself work wonders in clarifying a large number of formulas of plane and solid geometry ; without the scalar product the ideas of line and surface integrals and of Stokes's and Green's theorems are far from clear. The matter needs serious consideration and needs it now. We, as mathematicians,

must not continue indefinitely to hamper the instruction (alas, too rare) in theoretical physics by our negligence toward a very fruitful and suggestive field of our own science.

Gans starts Chapter I off with the statement that, in the neighborhood of an electric current, space has a certain condition called magnetic! He then proceeds systematically to develop this idea into the Maxwell equations $\text{curl } \mathbf{H} = 4\pi\mathbf{C}$. The second chapter treats induction in para- and diamagnetic bodies and results in the Maxwell equations $\text{curl } \mathbf{E} = -\dot{\mathbf{B}}$. It may be noted that Gans is using a different set of units from Schaefer. The magnetic potential is introduced and applied to several simple systems of currents. The boundary conditions in $\mathbf{B}$ and $\mathbf{H}$ are established and applied to numerous problems of a practical technical nature. The sphere, shell, prolate ellipsoid in a constant field are also treated. As this requires the introduction of curvilinear coordinates and Laplace's equation, it is clear that Gans is not quite so chary of his mathematics as Schaefer. But is it not a curious state of affairs and quite derogatory to the acumen and common sense and honesty of teachers of mathematics to think that curvilinear coordinates can find their place where vectors are afraid to enter? We go on teaching the same old things in the same old way, except that we may pay relatively more attention to rigor and relatively less to what is useful than in the old days. For instance, take a look at the last edition of Serret-Harnack-Scheffers's *Differential- und Integralrechnung* and see how it compares with Serret's original. See whether the new material and the amplifications have not to do with limits, convergence, and uniformity rather than with line and surface integrals, Gauss's, Green's, Stokes's theorems, and potential integrals.

Chapter III takes up ferromagnetic bodies. The start is made from the experimental curves of B to H and μ to H . The hysteresis diagram is discussed. Finally the equations

$$\nabla \cdot \mathbf{H} = 4\pi\rho, \quad \mathbf{H} = -\nabla\phi, \quad \nabla \cdot \nabla\phi = -4\pi\rho_0, \quad \phi = \iiint \frac{\rho_0}{r} dv$$

lead to the conception of the inverse-square law as applied to magnetism. If one looks real sharply he will find as a sub-heading in § 31 the mention of a magnetic pole! It is quite obvious that Gans is following out the Maxwell theory as indicated in the first sentences of the first chapter and that he is

not giving way in the slightest to the old point of view of magnetic poles and the law of the inverse square. This consistent attitude of the author is highly to be praised. In Chapter IV magnetic energy and force are treated with the aid of the Poynting vector. It is along toward the middle of this the last chapter in the tract that Gauss's methods of determining the magnetic moment of a magnet and the horizontal component of the earth's field are given. An especially commendable point is the analysis which leads up to the refutation of the common but false definition of para- and diamagnetic bodies by the position they assume relative to the lines of a magnetic field. Some mention of self- and mutual induction, Lenz's rule, and the ballistic galvanometer forms a close to the work.

It is indeed good to read a book like this of Gans where magnetism is treated systematically from the point of view of electromagnetism. The ordinary treatises on electromagnetism slip over the special field of magnetostatics with very little attention, the ordinary treatises on magnetism scarcely mention electromagnetism as other than a means and often they found electromagnetic effects on the study of magnetic shells. Now it may seem somewhat inconsistent in us to praise so heartily Schaefer's neat interrelating of action at a distance with action through a medium and still to support with almost equal ardor Gans's complete capitulation to the theory of a medium. But there is a vital difference between electrostatics and magnetostatics in that negative and positive electricity can be isolated, that is, $\text{div } \mathbf{D} \neq 0$, whereas a true magnetic pole is a fiction, that is, $\text{div } \mathbf{B} \equiv 0$. We recommend Gans to our readers as heartily as we recommended Schaefer; but don't begin with Gans. It is certainly true that Jahnke has made an auspicious start with his series of tracts; let us hope that he can keep up on the present high level; short, snappy, scientific, modern tracts are just what we need.

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BOSTON, MASS., *August*, 1910.

SHORTER NOTICES.

The Fundamental Laws of Addition and Multiplication in Elementary Algebra. By EDWARD V. HUNTINGTON, Harvard University. Reprinted from the *Annals of Mathematics*, Second Series, Volume 8, No. 1 (October, 1906). Publication Office of Harvard University. 44 pp.

THE object of this paper, as stated in the introduction, is "to present a list of fundamental propositions for algebra, from which, on the one hand, all the other propositions of algebra can be deduced, and in which, on the other hand, no superfluous items are included,—a list, in short, which is sufficient and free from redundancies."

The propositions are stated in terms of abstract undefined symbols a , b , etc., and two combinations of these symbols denoted by $\oplus$ and $\odot$. More briefly stated, the question which the paper answers is: "Given a class of elements with two rules of combination, what conditions must such a system satisfy in order to be formally equivalent to one of the systems of ordinary algebra?"

The first conditions imposed are the ten laws:

- A_1 . $a \oplus b$ is a unique element of the system.
- A_2 . $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.
- A_3 . (1) If $a \oplus x = a \oplus y$ then $x = y$.
(2) If $x \oplus a = y \oplus a$ then $x = y$.
- A_4 . If $\mu x = \mu y$ then $x = y$, μ being an ordinary integer.
- A_5 . $[a \oplus b = b \oplus a]$.
- M_1 . $a \odot b$ is a unique element of the system.
- M_2 . $(a \odot b) \odot c = a \odot (b \odot c)$.
- M_3 . (1) If $a \odot x = a \odot y$ and $a \oplus a \neq a$ then $x = y$.
(2) If $x \odot a = y \odot a$ and $a \oplus a \neq a$ then $x = y$.
- M_4 . (1) $a \odot (b \odot c) = (a \odot b) \oplus (a \odot c)$.
(2) $(b \oplus c) \odot a = (b \odot a) \oplus (c \odot a)$.
- M_5 . $a \odot b = b \odot a$.

Emphasis is placed on the purely abstract character of these symbols and on the formal character of the deduction from them. "How can we be sure that our deduction is rigorous? . . . The only way to avoid the danger of using in our reasoning other properties besides those expressly stated in the funda-

mental assumptions is to regard them not as axiomatic propositions about numbers, but as *blank forms* in which the letters a , b , c , etc. may denote any objects we please and the symbols $\oplus$ and $\odot$ any rules of combination; . . . The deduction from such blanks must necessarily be purely formal, and hence will not be affected by the troublesome connotations which would be sure to attach themselves to any concrete interpretation of the symbols." This we recognize of course as the general program of modern work in the foundations of mathematics.

The independence or non-redundancy of the system is exhibited in the usual manner, by showing concrete systems in which all but one of the assumptions are verified. Thus A_2 is shown to be independent of the other nine laws by the following systems: "Let the class considered be the class of all rational numbers. Let $a \oplus b = 2(a + b)$ and $a \odot b = ab$."

The following argument is given to show that A_5 is a consequence of the remaining nine of the ten fundamental postulates:

$$\begin{aligned}(a + b)(c + c) &= (a + b)c + (a + b)c \\ &= ac + bc + ac + bc \quad (M4_1, M4_2),\end{aligned}$$

also

$$\begin{aligned}(a + b)(c + c) &= a(c + c) + b(c + c) \\ &= ac + ac + bc + bc \quad (M4_3, M4_4).\end{aligned}$$

Hence

$$ac + ac + bc + bc = ac + bc + ac + bc$$

and

$$bc + ac = ac + bc \quad (A3_1, A3_2).$$

Hence

$$(b + a)c = (a + b)c \text{ and } b + a = a + b \quad (M4_5, M3_2).$$

The consistency of the fundamental laws is shown by giving a concrete example which is assumed to be self-consistent in which all the postulates are satisfied. The example chosen is the Argand diagram representing the complex numbers, in reality a vector field. It should be noted that this by no means proves absolutely the consistency of the postulates, but merely transfers the question to another realm. The consistency of each of the many special algebras developed in the paper follows from the fact that it is a subalgebra of this most general algebra employed.

It is shown that many essentially different systems satisfy these ten postulates. Thus they are all satisfied by the numbers

of ordinary algebra with addition and multiplication retaining their usual meaning, and also by a system in which $a \oplus b = a + b + 1$ and $a \odot b = ab + (a + b)$. This gives rise to a discussion of isomorphism and categoricity. Two systems satisfying these general laws are said to be isomorphic with respect to addition and multiplication when the following conditions are satisfied:

1) the elements of the two systems can be brought into one-to-one correspondence . . . and

2) this correspondence can be set up in such a way that whenever a and b in one class corresponds to a' and b' in the other class, then $a + b$ will correspond to $a' + b'$ and $a \times b$ will correspond to $a' \times b'$.

If a set of postulates is such that any two systems which satisfy it are isomorphic, then the set of postulates is said to be categorical.

While the ten postulates given above do not therefore form a categorical system, special laws may be added to them in various ways to form categorical systems. Thus by adding the postulates

E_1 . There is a unit element in the system.

F . There are no elements in the system besides those required by the other postulates.

It is then asserted that the algebra of positive integers is completely determined by the postulates $A_1, 2; M_1, 2, 3, 4; E_1; F$. In a paper avowedly abstract and purely deductive this seems a somewhat daring statement. It is proved, it is true, that these positive integral elements form a closed system with respect to addition and multiplication, but is this equivalent to showing that the set of postulates just given is categorical according to the definition above? The truth seems to be that we are certain our system is as categorical as some other systems to which we may prove it equivalent. Again, as with the question of consistency, the difficulty has simply been transferred to another realm.

Many other categorical systems are obtained in a manner similar to the above by adding special assumptions to those already given.

It should be remarked that in order to form an independent system each of the ten fundamental postulates except A_1 and M_1 must be regarded as making the existence of the sums and products hypothetical. Thus A_2 must read, if stated fully, "If $(a \oplus b) \oplus c$ and $a \oplus (b \oplus c)$ exist, etc."

The original name for the unproved propositions of a mathematical science was "axiom," — a truth so simple that everyone must assent to it whenever the statement is fully comprehended. In this respect the point of view has changed completely. If a , b , $\oplus$, $\odot$ are purely abstract symbols, then no proposition whatever is evident about them. Hence the word "axiom" with its old connotation is being discarded. The paper under review uses "postulate." Other writers, as Veblen and Young, are using "assumption."

This paper should serve two distinct and very useful purposes. The writer of elementary algebras for college use will have at hand a set of postulates which will serve directly as a basis for much of his work and as a model which we hope may guide his way in making the extensions necessary to characterize the complete algebra. This should render less prevalent in the future the numerous logical incongruities that so often have marred otherwise excellent texts.

The subject with which the paper deals is confessedly abstract but the style is so lucid and the mode of treatment so simple that it should be within the reach of students even in the first years in college. It is the feeling of the reviewer that the reader who is to take his first dip into abstract mathematics cannot very well do better than to read this elegant introduction to the logical foundations of algebra.

N. J. LENNES.

Leçons sur la Théorie de la Croissance. By ÉMILE BOREL.
Paris, Gauthier-Villars, 1910. vi + 168 pp.

THIS book is one of the excellent series of monographs on the theory of functions appearing under the general editorial direction of M. Borel. It has grown out of two independent courses of lectures on the theory of increase (croissance) delivered by the author at the University of Paris during the winter semesters of 1907–1908 and 1908–1909 respectively. These two courses have been coordinated and unified by M. Arnaud Denjoy.

A function $f(x)$ is said to be increasing if $f(x') - f(x'')$ is positive when $x' - x''$ is positive. The theory of increase is devoted to the investigation of the rate of change of an increasing function $f(x)$ with respect to x as x approaches infinity.

The author begins his preface by giving expression to his growing conviction that the theory of increase is the essential

basis of the theory of functions. But it seems to him preferable to wait for some years before undertaking a systematic theory of functions in which the theory of increase would serve as an introduction the essential results of which would be constantly called into use. This mode of exposition will bring about a considerable simplification; but it requires the employment of new terms and new notations, and these should be introduced with a great deal of care.

Besides the introduction (13 pages) the book contains five chapters, of which the first three are devoted to the general theory of increase and the last two to its applications.

In Chapter I (19 pages) certain fundamental types of increasing functions are considered and with each is associated a sign which is used to denote its order of increase; for instance, the orders of x^n and x^ω are said to be n and ω respectively. Operations of addition and multiplication of orders of increase are defined and their laws are developed.

In Chapter II (9 pages) the consideration of approximate orders of increase is taken up. The function

$$f(x) = x^n (\log x)^{2 + \sin x},$$

for instance, is said to be of order (n) (read " n parenthesis"). The use of (n) in this case to denote the approximate order of $f(x)$ is justified by the fact that the order of magnitude of $f(x)$ is less than that of $x^{n+\epsilon}$ and greater than that of $x^{n-\epsilon}$, where ϵ is any positive quantity. The laws of addition and multiplication of approximate orders are investigated.

The third chapter (32 pages) is devoted to a study of the changes produced in the order of increase of a function by the integration and differentiation of the function.

Chapter IV (44 pages) on analytical applications falls into three parts. In the first part series with constant positive terms are considered. If such a series diverges, the sum of its first n terms, which increases indefinitely with n , possesses, with respect to n , an order of increase which it is useful to know. If the series is convergent one can propose the problem of determining the order of decrease of the remainder after n terms. Both of these questions are treated. Preparatory to a corresponding investigation for infinite products and as contributing to it, the author in the second part of the chapter develops the fundamental properties of the gamma function.

They are deduced in a very simple way from the preceding general theory. Finally, the third part of the chapter is given over to the study of infinite products and especially to a comparison of the increase of functions with that of their zeros.

The last chapter (51 pages) is devoted to arithmetical applications. Its object is to utilize the theory of increase in the classification of incommensurable numbers. An indication of the contents of the chapter is given in the following list of topics: Relations between incommensurable numbers and increasing functions; continued fractions; approximation by means of rational numbers to certain classes of numbers; numbers of the second order and periodic continued fractions; approximation by means of rational numbers to any numbers whatever; approximation by means of algebraic numbers; case of the number e .

In this little book M. Borel has rendered valuable service by emphasizing in an effective way the fact that the theory of increase has a place of growing importance in the development of the theory of functions.

R. D. CARMICHAEL.

Elemente der Mathematik. By JULES TANNERY. Autorisierte deutsche Ausgabe von P. KLAESS. Leipzig, Teubner, 1909. x + 339 pp. 8 marks.

Two objects dominate the teaching of freshman mathematics: the one, which may not too inaccurately be called the classical, endeavors to give the student a certain proficiency in algebraic analysis or in geometrical logic; the other, which may be called the psychological, endeavors to give him a wide view over the realm of mathematics. The classical has for its educative end skill; the psychological, vision. The classical endeavors to build the structure by finishing one room at a time. The psychological endeavors to build first the framework and then to fill in the details. Whether it is better for the great mass of freshmen, who will have little or no mathematics after that year, to learn well how to work Horner's process or to solve trigonometric equations, or whether there should be preferred a knowledge of "what is a function, its variation, and the representation of its values," or a knowledge of how we determine "tangents, areas, volumes, and the new functions, curves, or surfaces thus arising" — is a question now in

process of solution. To further the new view of what is desirable as it has been planned for French schools, the book under review was written.

The introduction of over one hundred pages clears up the notions of the student acquired in his secondary school work. These are usually vague and half-forgotten, and in most cases in such state that the student "has no desire to continue nor does he see any use for what he has already learned." After this review of first principles, the real advance begins and proceeds in a delightfully graceful manner from the consideration of the important conclusions that may be drawn from elementary identities, into the rise and solution of the quadratic equation; thence into coordinates, functions, limits, derivatives, and integrals. The intuition is constantly appealed to and nowhere is there difficulty for any student to follow the easy grade the course pursues. The last chapter, on astronomy, leaves the student face to face with one of the boundless realms he may enter through the mathematical door.

One can but feel that a student who has had such a course as this will have more genuine love and enthusiasm for continuing his mathematical studies than one who has been drilled in the conventional way. The original (or this translation or, let us hope, an English translation) ought to be in the hands of at least every teacher of elementary mathematics.

JAMES BYRNE SHAW.

Komplex-Symbolik. Eine Einführung in die analytische Geometrie mehrdimensionaler Räume. Von ROLAND WEITZENBÖCK. Leipzig, Göschen (Sammlung Schubert, LVII), 1908. 191 pp.

THE real subject of this little volume is line geometry in three and higher dimensions. The treatment however is not according to the old familiar methods but by the use of symbolic notation. The symbolism is founded on that of Clebsch, with which the author states that the reader should be familiar in order to follow this book with ease. The reviewer also wishes to emphasize this need of reading something on symbolic notation before undertaking this book: The symbolism however is sufficiently developed so that one can readily understand it, but a previous knowledge makes the reading more enjoyable. The only other attempt known to the reviewer to treat the subject of line geometry symbolically is that of E. Waelsch ("Zur Invariant-

entheorie der Liniengeometrie," *Sitzungsberichte der k. Akademie der Wissenschaften*, Wien, 1889). The author states that he developed the subject without knowledge of Waelsch's work so that there naturally is divergence of treatment.

In many places the author has not made his meaning clear, and this coupled with frequent typographical errors makes the reading rather difficult. But after one has gone through with it the point of view gained is well worth the trouble.

Two kinds of symbols are defined, viz., ordinary and complex. Ordinary symbols are those which obey the commutative law of multiplication, complex symbols are those which do not. The ordinary coefficients are of the first kind, and the symbols used to represent the line coordinates are of the second kind. Thus for line coordinates we have

$$p_i p_k = p_{ik} = -p_{ki} = -p_k p_i.$$

The quantities p_i and p_k have no meaning except as symbols. After these definitions the general properties of complex symbols are discussed and applied to the linear and quadratic complex in three dimensions and to finding the invariants, covariants, and contravariants of systems of lines.

Then follows the discussion of linear systems of lines in higher dimensions. Here the author has contributed much new material. After reading the discussion of the linear complex in s_4 one appreciates how much more direct and simple is the treatment of the same subject by Castelnuovo. But nevertheless after the symbolism is built up it enables one to see much that might otherwise escape him.

Throughout the book the author has made good use of the idea of defining a line, curve, or in higher dimensions, a plane, etc., by the system of lines which cut it. This has a decided advantage for certain problems, since a single equation then represents the line, curve, etc.

The book closes with an excellent chapter outlining a general symbolic analytic geometry and setting forth some of its advantages.

C. L. E. MOORE.

Analytische Geometrie des Punktpaares, des Kegelschnittes und der Fläche zweiter Ordnung. Zweiter Teilband: Von Dr. OTTO STAUDE. Leipzig and Berlin, Teubner, 1910. iv + 452 pp., with 47 figures.

THIS volume continues the work reviewed in the October (1910) BULLETIN, the page and paragraph numbering being continuous, and the entire work containing one thousand pages. The valuable bibliography and notes relating to the whole work cover sixty pages at the end of the present volume. Since the character of the treatment of the various topics is the same as that in the earlier volumes, little comment seems necessary.

After classifying surfaces of the second class in various ways, the author devotes nearly one hundred pages to a detailed study of plane sections of surfaces of the second order. Circular and equilateral hyperbolic sections receive particular attention. The second part, containing three chapters, is devoted to confocal systems of quadrics, and to general focal properties, both of the surfaces and of their plane sections. In the first chapter, by regarding x, y, z , as homogeneous coordinates of the lines of a sheaf on a point O , instead of ordinary point coordinates in space, the properties of confocal systems of cones are very neatly established. The so-called elliptic coordinates of the lines of a sheaf, and of the points of space are explained and used to some extent in subsequent proofs. The axis complexes of confocal systems are discussed and a large number of theorems relating to confocal systems proven. The second chapter is devoted to the Amiot, the MacCullagh, and the Jacobi focal properties. To any one not familiar with the theorems, this chapter will prove interesting reading. The third chapter on broken focal distances contains a large number of theorems and formulas relating to metrical relations existing between the points of the focal ellipse and the points of the corresponding focal hyperbola, and between the points of two conjugate focal parabolas. From these focal properties, the "thread" construction of the ellipsoid is deduced, and the corresponding property and the construction of the ellipse on the plane are shown analytically to be a special case of this.

The third part consists of two long chapters on surfaces of the second order and of the second class in tetrahedral coordinates. In the first chapter surfaces are again classified, conditions for degeneracy stated, plane sections discussed, etc., as previously done in ordinary coordinates. The complex of tangents to a surface and the six generator complexes are studied. The theorem is established that the generators fall into two systems, each system being composed of the lines common to three of the six linear generator complexes. These six com-

plexes are thus divided into two groups of three each. The last chapter is devoted largely to polar properties. The orthogonal linear substitutions corresponding to the real polar tetraedra; the properties of tetraedra determined by four generators, two from each system; and the Pascal and Brianchon theorems in space are other topics in this chapter.

This work with its extensive bibliography, and hundreds upon hundreds of references to the literature on the subject, must represent a prodigious amount of labor and pains on the part of the author. As an encyclopaedic reference book it will undoubtedly be useful, although the index does not seem very satisfactory. For example, twenty or more pages are devoted to a discussion and comparison of the Amiot and MacCullagh focal properties, yet neither of these names appears in the index.

D. D. LEIB.

Festschrift zur Feier des 100 Geburtstages Eduard Kummers mit Briefen an seine Mutter und an Leopold Kronecker. Herausgegeben vom Vorstande der Berliner Mathematischen Gesellschaft. Leipzig, Teubner, 1910. 103 pp.

It was very appropriate that the memorial address on Kummer should be delivered by K. Hensel, whose remarkable investigations on algebraic numbers entitle him to speak with authority on Kummer's chief triumph, the creation of ideal numbers. We find here an elementary and entertaining account of Kummer's early interest in Fermat's equation $x^\lambda + y^\lambda + z^\lambda = 0$, where λ is a prime, which led him to investigate the numbers $a + b\alpha + \dots + k\alpha^{\lambda-1}$, where α is a complex λ -th root of unity, and $a, b, \dots, k$ are integers. It appears on excellent authority that Kummer at an early period supposed that he had a complete proof of the impossibility of Fermat's equation and laid before Dirichlet a manuscript purporting to give such a proof. The latter pointed out to Kummer that, although he had proved that any number $f(\alpha)$ was the product of indecomposable factors, he had assumed that such a factorization was unique, whereas this was not true in general. After years of study, Kummer concluded that this chaotic state of affairs following from the non-uniqueness of factorization was due to the fact that the domain of the numbers $f(\alpha)$ was too small to permit the presence in it of the true prime numbers, and was led to his epoch-making creation of ideal numbers. Without attempting to explain the elaborate machinery of Kummer's

method, so delicate that an expert must handle it with the greatest care, and nowadays chiefly of historical interest in view of the simpler and more general theory of Dedekind, Hensel contented himself in his address with a luminous elementary account of the essential features of Kummer's method as developed for a special set of integers. It should be stated that, while it is preferable to take the newer standpoint as regards the foundation of the theory of ideals, this change of viewpoint has not altered the validity of the rich array of fundamental results obtained by Kummer. The brief sketch on page 30 of Kummer's method of proving the impossibility of Fermat's equation, when λ is a regular prime, applies directly only to the first of the two cases into which Kummer divided the discussion. However, the same use of the class number is employed in the second case and this point is the one being emphasized by Hensel. All admirers of Kummer will take keen pleasure in reading this masterly Gedächtnisrede.

The 56 pages of letters from Kummer to his favorite and most gifted pupil Kronecker are of historic value. They give a first-hand view of the progress step by step made by Kummer in his construction of his own imperishable monument.

L. E. DICKSON.

Interpolationsrechnung. Von T. N. THIELE, Em. Professor der Astronomie an der Kopenhagener Universität, Präsident des Vereins Dänischer Aktuari. Leipzig, Teubner, 1909. xii + 175 pp.

In this book the theory of interpolation is developed so as to emphasize its role in pure mathematics as well as its place in the calculations of applied mathematics. The book is divided into four chapters. The elementary treatment in the first chapter is based on the general interpolation formula of Newton, written

$$(1) \quad X = A + (x - a)[\delta(a, b) + (x - b)\{\delta''(a, \dots, c) + (x - c)\{\delta'''(a, \dots, d) + (x - d)\delta''''(x, a, \dots, d)\}\}],$$

where $a, b, c, d, \dots$ are distinct values of the argument, $A, B, C, D, \dots$ are corresponding tabulated values, and $\delta(a, b), \delta''(a, \dots, c), \dots$ are "divided differences" defined by

$$\delta(a, b) = \frac{A - B}{a - b}, \quad \delta''(a, \dots, c) = \frac{\delta(a, b) - \delta(b, c)}{a - c}, \dots$$

A number of theorems are proved concerning divided differences, and it is shown, by numerical applications, that formula (1) is easily usable if the arguments be chosen in certain ways. After expressing the derivative of the n th order $d^n X/dx^n$ in terms of divided differences with repeated arguments, the first application is to develop Taylor's series in terms of divided differences. The next application is made to the solution of numerical equations. The coefficients are expressed as divided differences and it is maintained that the interpolation method is one of the best methods of finding the real roots of numerical equations.

Interpolation by means of infinite series is treated to provide for cases where divided differences of no finite order have the value zero. The possibility that a Newton's interpolation formula, even if convergent, does not represent the function in question, is considered. The author maintains, however, that the functions of the most importance in applied mathematics, fortunately, belong to a class that can be interpolated; at least within certain limitations on the argument. He states the theorem that, for this class of functions, the interpolated values coincide with the function in question if for an infinite number of values of the argument, in a finite interval, there is exact coincidence. It seems that a characterization of functions as belonging to applied mathematics is not definite, in the sense that it is a basis for proving a theorem, and it should be emphasized that the method by which the theorem is established involves questions of convergence.

The first chapter ends with a careful investigation into the representation of a^x by a method of interpolation.

The second chapter is devoted to symbolic representation; the rather burdensome notation is made readable by the presentation of numerous numerical examples.

The third chapter treats the question of interpolation when the function has singularities, and presents graphic interpolation as an auxiliary method. Near the end of the third chapter, interpolation by reciprocal differences is presented. By means of these reciprocal differences, there arises a general method of interpolation that involves continued fractions and that is entirely analogous to the method of Newton. The general formula for interpolation by reciprocal differences is given. It is indicated that the precise applicability of reciprocal differences to interpolation is much more extensive, in a certain sense, than

that of divided differences. This arises from the fact that where the method by divided differences is limited to integral functions, the method of reciprocal differences is limited to the most general form of fractional functions. To be sure, the application of reciprocal differences is also much more difficult than that of divided differences.

In the fourth chapter interpolation for functions of two or more variables is presented in an interesting and useful manner.

The book, as a whole, is a scholarly and readable presentation of the elements of the calculus of finite differences, and should be found of value not only to those interested in the arithmetical application of interpolation, but also to those interested in a theoretical treatment of the subject.

H. L. RIETZ.

Plane Geometry, with Problems and Applications. By H. E. SLAUGHT and N. J. LENNES. Allyn and Bacon, Boston, 1910. vi + 280 pp.

THIS book has several features that distinguish it from the conventional high school text. Among these the most noticeable are the gradual introduction of the severely logical forms, and the introduction, for the purpose of making the subject more attractive, of a large number of applications to geometric forms more or less commonly met with in life.

The book is divided into seven chapters, the first five of which correspond in a general way to Books I to V of Euclid, except that certain of the more difficult theorems, and the subject of incommensurable ratios are deferred to the last two chapters. Chapter I begins as usual with an introduction containing the common definitions. Numerical equality and geometric equality, or congruence, are sharply distinguished. The first propositions and problems are then introduced (pages 14-25) in an informal way. Then a few axioms are stated formally, and a number of theorems are given as "preliminary theorems," some of which are easy consequences, and some of which, for the purposes of the text, are assumed. A general discussion on the nature of a demonstration follows, after which proofs are given in the usual form. Aside from the introduction of the applications and the deferring of the matter indicated above, the content of the first five chapters is about that of the usual book. The algebraic form of the treatment is a decided improvement. One particularly pleasing feature is the willing-

ness of the authors to introduce assumptions, as such, wherever they deem it desirable. At the same time, they state frankly that the set of axioms given is not complete, and that intuitional inferences concerning such subjects as order and continuity are drawn freely.

Chapter VI contains the treatment of variable geometric magnitudes, the subject being treated from the standpoint of the graph, a new feature which should aid in clearing up this subject.

Chapter VII contains general remarks on the nature of axioms, proofs, definitions, etc., from the strictly logical point of view; problems on loci, proofs of the incommensurable cases in the proportion theorems, mean and extreme ratio, area of a triangle in terms of its sides, and a few problems on maxima and minima. Length of arc and area of a circle are defined as limits.

The applications, which occur in large numbers throughout the book, are usually taken from architectural designs and various ornamental designs for decorative purposes, such as tile patterns, parquet floors, grill work, etc. These exercises are for the most part very simple, but by bringing into play a large number of straight lines and circle arcs in a single problem, they are a valuable aid in the development of geometric imagination. These exercises also bring into bold relief the principle of symmetry, which is thereby given the prominence that it deserves. Other applications are made to the problems of finding the distance between two points on opposite sides of a pond, measuring the height of a tree, finding the distance between two inaccessible points, cutting the braces for a roof, etc. All these applications should be a valuable aid in holding the interest of the numerous pupils who care comparatively little for the logical features of the subject. An ordinary class would probably find more than enough material of this kind in the book, but the omission of a part of it would in no way affect the continuity of the subject.

A novel feature is the introduction of the sine of an angle, with a few exercises in its use (pages 136-139).

The book as a whole appears to be a very teachable text. The typographical work is good, and the figures are clearly drawn. As compared with the ordinary text, the book has lost nothing and gained much by the innovations.

F. W. OWENS.

Shop Problems in Mathematics. By W. E. BRECKENRIDGE, S. F. MERSERAU, and C. F. MOORE. Boston, Ginn and Company, 1910. vii + 280 pp.

INTENDED for use in trade schools, the book is divided into two distinct portions, the earlier and major part consisting essentially of problems from the pattern shop, forge, foundry, and machine shop; and the remainder in the presentation of a review of certain parts of elementary mathematics which are involved in the earlier problems. Demonstrations are not presented, results only being desired. A chapter on the slide rule is also included, in which an error in statement is implied which may lead the reader to believe that much of slide rule work is exact. The volume gives the impression of a lack of organized sequence in the arrangement of the material in the individual chapters.

C. F. CRAIG.

The Integrals of Mechanics. By O. C. LESTER. Boston, Ginn and Company, 1909. vi + 67 pp.

THE author has collected many of the integrals found in theoretical mechanics and presents them with very little physical interpretation. After an introductory chapter reviewing the applications of the integral calculus to lengths of arc, area, etc., the subjects of center of gravity, moment of inertia, and ellipsoids of inertia are treated in order. Stress is laid on the presentation of formulas for many of the special cases and on theorems derivable from special examples. The general formulas as well as the special ones are obtained by what are at best only semi-rigorous methods. Although the volume is intended as an introduction to theoretical mechanics, most of the material seems usable as an introduction to applied mechanics.

C. F. CRAIG.

Annuaire du Bureau des Longitudes pour l'An 1911. Paris, Gauthier-Villars.

SEVERAL improvements are noticeable in the issue for the current year. The section on the classification of stellar spectra has been rewritten by M. de Gramont; the elements of the moon and major planets are referred to the epoch 1900 instead of 1850; and M. Schulhof contributes a résumé of the information obtained during the last two years concerning Halley's

comet and the comets of 1909. Tables of variable stars have been dropped because the number has become too great for record in a volume of this kind. Minor improvements have also been made in the sections: geography and statistics, coinage, weights and measures, and meteorology.

The appendices contain an account of the sixteenth meeting of the international geodetic association written by M. Poincaré in his well-known luminous and entertaining style. The eclipse 1912 April 17 is treated by M. G. Bigourdan. The central line passes across France; unfortunately the calculated maximum duration of totality is only six seconds and it is doubtful whether even this small interval will be attained. In certain parts the eclipse will be annular, in others total. Obituary notices of Bouquet de la Grye and Paul Gautier are contributed by MM. Poincaré and Baillaud.

ERNEST W. BROWN.

Leçons élémentaires sur le Calcul des Probabilités. . Par R. DE MONTESSUS. Paris, Gauthier-Villars, 1908. vi + 191 pp.

THERE is perhaps not much chance for striking novelties, whether of material or of arrangement, in elementary texts on the theory of probabilities; the general model, especially in France, seems deservedly to be Bertrand's excellent *Calcul des Probabilités*. The present author has on numerous occasions manifested his interest in various questions concerning probability, and in particular in regard to the proper definition of chance; he does not approve of the scheme of founding the laws of chance upon ignorance as to what may happen and upon the law of sufficient reason, but he believes rather that the laws should be developed out of the experience that in the long run certain things do happen in a particular way for a definite percentage of the total number of ways they may happen. This seems to us, as to the author, somewhat more satisfactory than the older method; it is better to say that in tossing coins heads are found to come half the time and hence the probability of heads is one half, rather than to say that there is no reason why heads should fall rather than tails and hence heads will fall half the time.

Not only in his introductory remarks but throughout the book the author discloses philosophical as well as mathematical tendencies. Moreover, the subjects which he touches are numerous; it is not usual to find mention of insurance and of

speculation in addition to the discussion of various games of chance and other such canonical subjects for treatises on probabilities. As many topics are touched upon, they can naturally not all be more than merely touched, and the author has been good enough to indicate many references where the different subjects may be further pursued. The elementary character, the clear style, the varied topics, the careful references, all combine to make the work useful and thoroughly to be recommended to a wide range of readers who have some, not necessarily much, knowledge of mathematics.

E. B. WILSON.

NOTES.

THE April meeting of the AMERICAN MATHEMATICAL SOCIETY will be held at the University of Chicago on Friday and Saturday, April 28-29. At this meeting Professor MAXIME BÔCHER will deliver his Presidential Address, the provisional title of which is: "Charles Sturm's Published and Unpublished Work on Differential and Algebraic Equations." Except for the summer meetings, this will be the first united meeting of the whole Society since 1896. A large attendance is expected from all sections of the country. Titles and abstracts of papers to be presented at this meeting should be sent to the Secretary of the Society at an early date.

THE January number (volume 12, number 2) of the *Annals of Mathematics* contains the following papers: "Rationality groups in prescribed domains," by S. EPSTEIN; "Envelopes of one parameter families of plane curves," by W. J. RISLEY and W. E. MACDONALD.

AT the meeting of the London mathematical society held on February 9 the following papers were read: By E. CUNNINGHAM, "The application of the mathematical theory of relativity to the electron theory of matter"; by G. B. MATHEWS and W. E. H. BERWICK, "The reduction of arithmetic binary forms which have a negative discriminant"; by H. BATEMAN, "Certain vectors associated with an electromagnetic field and the reflection of light at the surface of a perfect conductor."

THE royal academy of sciences of Turin announces the following conditions for the Vallauri prize: "A prize will be awarded by the Academy, without distinction of nationality, to the author; who, in the interval from January 1, 1915, to December 31, 1918, shall have published the most extensive and celebrated work in the domain of the physical sciences, the word being understood in its broadest sense. The amount of the prize is twenty-six thousand lire, to be paid one year after being awarded. Members of the Academy may not compete. No notice will be taken of memoirs in manuscript."

THE royal academy of sciences of Bologna announces the following prize problem: "To investigate by critical and historical methods the organic development of the theory of elliptic functions and the various points of view under which the theory has been considered from the end of the eighteenth century until the present. Indicate the influence which these various views have had upon other branches of analysis." Competing memoirs should be plainly written in Italian and sent, under the usual conditions, to the secretary of the academy before December 31, 1912.

THE Dutch scientific society announces the following prize problems, the award to be made January 1, 1912:

1. It is required to determine the prime numbers p which satisfy the congruence $g^{p-1} - 1 \equiv 0 \pmod{p^a}$, in which g and a are given integers larger than unity.

2. The society desires an experimental and theoretical study of the phenomena of critical opalescence, either in gases or in mixed liquids, and the particularities in the characteristic equation which can be attributed to the same causes as critical opalescence.

Competing memoirs should be sent, under the usual conditions, to the secretary, Dr. J. P. Lotsy, in Harlem. The prize is either a gold medal or a cash award of 150 florins, in the option of the successful competitor. In the case of a memoir of exceptional value, an extra sum of 150 florins may also be awarded.

THE Macmillan Company announces the following work as in press, to appear in a few weeks: *Fundamental Concepts of Algebra and Geometry*, by Professor J. W. YOUNG, of the University of Kansas.

THE next annual meeting of the British association for the advancement of science, will be held at Portsmouth, beginning August 30, under the presidency of Sir WILLIAM RAMSEY; Professor H. H. TURNER is chairman of the section of mathematical and physical sciences.

BEGINNING with volume 13 (1911), the *Enseignement Mathématique* will be published jointly by Gauthier-Villars in Paris, Georg and Co. in Geneva, and Teubner in Leipzig.

THE following doctorates in mathematics were conferred by the German universities during the academic year 1909-10:

Breslau.

DITTRICH, R. "Abstandsörter im Polarraume."

JURETZKA, E. "Die Entwicklung unstetiger Funktionen nach den Eigenfunktionen des schwingenden Stabes auf Grund der Theorie der Integralgleichungen."

NEUMANN, NELLY. "Ueber das Flächennetz 2. Ordnung und seine korrelative Beziehung auf einen Strahlenbündel."

Erlangen.

BALDUS, R. "Über Strahlensysteme, welche unendlich viele Regelflächen 2. Grades enthalten."

GERSTENMEIER, C. "Beiträge zur Theorie der linearen Differentialgleichungen mit 4 und 5 singulären Stellen."

GEUS, A. "Die eindeutigen Transformationen der ebenen Kurve dritter Ordnung in sich, invarianten- und funktionentheoretisch behandelt."

Giessen.

WEHRHEIM, H. "Über das kombinatorische Produkt dreier Kollineationen in der Ebene."

Göttingen.

BIEBERBACH, L. "Zur Theorie der automorphen Funktionen."

COURANT, R. "Ueber die Anwendung des Dirichletschen Prinzips auf die Probleme der konformen Abbildung."

FREUNDLICH, E. "Analytische Funktionen mit beliebig vorgeschriebenem unendlich-blättrigem Existenzbereiche."

JAEGER, M. "Graphische Integrationen in der Hydrodynamik."

KAHN, GRETE. "Eine allgemeine Methode zur Untersuchung der Gestalten algebraischer Kurven."

LÖBENSTEIN, KLARA. "Ueber den Satz, dass eine ebene, algebraische Kurve 6. Ordnung mit 11 sich einander ausschliessenden Ovalen nicht existiert."

WINK, A. "Über die Diskontinuitätsbereiche der Gruppen aus linearen nicht infinitesimalen Substitutionen."

Greifswald.

BEYER, T. "Die Integration der simultanen linearen Differentialgleichungen mit konstanten Koeffizienten."

Halle.

LEHMANN, P. "Beiträge zur Theorie der Darstellung der stetigen Funktionen durch Reihen von ganzen rationalen Funktionen."

ROESER, E. "Die Verfolgungskurve auf der Kugel."

Heidelberg.

CARLEBACH, J. "Lewi ben Gerson als Mathematiker."

Kiel.

KOCH, W. "Beiträge zur affinen Geometrie der Flächen zweiten Grades."

Königsberg.

GAEDECKE, W. "Die inversen Flächen der Mittelpunktsflächen 2. Ordnung."

SCHIMANSKI, E. "Die algebraischen Invarianten der projektiven Gruppen der Ebene und die geometrische Charakterisierung dieser Gruppen."

Leipzig.

ALBERTI, H. "Die Grundlagen des Systems Spinozas in Lichte der kritischen Philosophie und der modernen Mathematik."

FEYER, W. "Über die Höldersche Funktion

$$Fu = e^u \cdot \prod_{n=-\infty}^{+\infty} \{ (1 - u/n)^n \cdot e^{u + \frac{1}{2}u^2/n} \}$$

und einige verwandte Transcendente."

München.

ASHTON, C. H. "Die Heineschen *O*-Funktionen und ihre Anwendungen auf die elliptischen Funktionen."

LOEHL, A. "Über konforme und äquivalente Transformationen im Raum. Ein Beitrag zur Geometrie der Kugeln."

ROSENTHAL, A. "Untersuchungen über gleichflächige Polyeder."

Münster.

FERRARI, F. "Die geometrische Lösung der Aufgaben dritten und vierten Grades mittels des Lineals und einer festen Kurve dritter Ordnung mit Rückkehrpunkt oder reellem Doppelpunkte."

Rostock.

DÜKER, W. "Über Beziehungen der Strahlenkomplexe zweiten Grades zu den Flächen zweiter Ordnung."

NADLER, C. "Über den Zusammenhang der Raumkurve vierter Ordnung erster Spezies mit ihrem Polartetraeder."

WOLFF, H. "Behandlung des Vorganges, dass eine ebene elektromagnetische Welle, die auf die ebene Oberfläche eines Körpers, insbesondere eines Leiters auftrifft, von diesem reflektiert wird, auf Grund der Maxwellschen Gleichungen unter ausführlichen Eingehen auf die Art der stattfindenden Energiefortpflanzung."

Strassburg.

BRAUN, WANDA. "Bestimmung der Körperdiskriminante in einem kubischen Zahlkörper."

BURGWEDEL, R. "Über die Eulerschen und Gauss'schen Methoden der Primzahlbestimmung."

GIROD, L. "Das sphärische Analogon der Hypocykloidenbewegung des Cardanus und sein Zusammenhang mit der Theorie eines verallgemeinerten Hooke'schen Gelenkes."

KIEFER, A. "Die Einführung der homogenen Koordinaten durch K. W. Feuerbach."

PLATE, H. "Punktausgleichung und Fehlerbestimmung nach graphischen Methoden in ihrer Anwendung auf Ortsbestimmung durch Standlinien."

STAMPFLI, O. "Der Zweiteilungskörper der elliptischen Funktionen."

Tübingen.

BLUM, F. "Die infinitesimale Biegung von Flächen bei vollständiger Starrheit eines Kurvensystems."

Würzburg.

GRÄBNER, G. "Algebraische Bertrand-Kurven und algebraische Kurven konstanter Torsion."

DR. W. SIERPINSKI, of the University of Lemberg, has been promoted to an associate professorship of mathematics.

PROFESSOR A. SOMMERFELD, of the University of Munich, has been elected a member of the Bavarian academy of sciences.

PROFESSOR P. STÄCKEL, of the technical school at Karlsruhe, has been elected to membership in the Heidelberg academy of sciences.

PROFESSOR M. PASCH, of the University of Giessen, will retire at the end of the present semester.

PROFESSOR W. F. MEYER, of the University of Königsberg, has received the title of Geheimer Regierungsrat.

PROFESSOR L. SCHLESINGER, of the University of Klausenburg, has accepted the professorship of mathematics at the University of Budapest.

PROFESSOR H. W. KUHN, of the University of Ohio, has been granted leave of absence during the second half of the present academic year, to study in Europe.

DR. ANNA J. PELL has been appointed instructor in mathematics at Mount Holyoke College.

PROFESSOR A. W. PHILLIPS, of Yale University, will retire from active service at the close of the present academic year. Professor Phillips has taught in Yale College since 1876, becoming professor of mathematics in 1891. Since 1895 he has been the dean of the graduate school.

MR. C. J. WHITE, professor of mathematics at Harvard University from 1885 to 1894, has been appointed professor emeritus.

PROFESSOR CHARLES MÉRAY, of the University of Dijon, died February 7, at the age of 75 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BARBETTE (E.). Le dernier théorème de Fermat. Liège, Gnuse. 8vo. 19 pp.

BERZOLARI (L.). Geometria analitica, I: Il metodo delle coordinate. (Collection Manuali Hoepli.) Milan, Hoepli. 16mo. 409 pp. L. 3.00

FABRY (E.). Théorie des séries à termes constants. Paris, Hermann, 1910. 8vo. 200 pp. Fr. 6.50

FESTSKRIFT, H. G. ZEUTHEN. Fra venner og elever i anledning af hans 70 aars fødselsdag. Kjobenhavn, Host, 1910. 8vo. 166 pp.

FREUNDLICH (E.). Analytische Funktionen mit beliebig vorgeschriebenen unendlich-blättrigen Existenzbereichen. (Diss.) Gottingen, 1910. 8vo. 40 pp.

FRICKE (R.) und KLEIN (F.). Vorlesungen über die Theorie der automorphen Funktionen. 2ter Band: Die funktionentheoretischen Ausführungen und die Anwendungen. 2te Lieferung: Kontinuitätsbetrachtungen im Gebiete der Hauptkreisgruppen. Leipzig, Teubner, 1911. 8vo. pp. 283-438. M. 7.00

HUYGENS (C.). Tome XII: Travaux de mathématiques pures. Paris, Gauthier-Villars. Fr. 20.00

JÜTTE (O.). Die Schmiegunskugel einer Flächenkurve. (Diss.) Halle, 1910. 8vo. 8+143 pp.

KLEIN (F.). See FRICKE (R.).

KNESER (A.). Die Integralgleichungen und ihre Anwendungen in der mathematischen Physik. Vorlesungen an der Universität zu Breslau gehalten. Braunschweig, Vieweg, 1911. 8vo. 8+243 pp. M. 7.00

LEHMANN (F. J.). Harmonic analysis. Oberlin, Comings, 1910. 8vo. 9+156 pp. \$1.50

MEDUGNO (M.). Della trasformazione birazionale [2-4]. Napoli, Accademia delle Scienze, 1910. 8vo. 188 pp.

MERKENS (E.). Ueber gewisse räumliche Punktmengen, die sich als stetige Flächen auffassen lassen. (Diss.) Königsberg, 1910. 8vo. 86 pp.

SPANO (D.). Sulla identità degli sviluppi in serie del Wronski e del Lagrange. Napoli, Accademia delle Scienze, 1910. 8vo. 10 pp.

VIVANTI (G.). Lezioni di analisi infinitesimale. Puntata I. Pavia, Mattei, 1910. 8vo. 304 pp. L. 15.00

VOIGT (A.). *Theorie der Zahlenreihen und der Ringgleichungen.* Leipzig, Göschen, 1911. 8vo. 8+133 pp. M. 4.00

II. ELEMENTARY MATHEMATICS.

AMALDI (U.). See ENRIQUES (F.).

ANZLINGER. *Algebraische Aufgaben mit vollständiger Lösung durch einfache Schlüsse.* Buhl, Baden, Konkordia A.-G.

ARMENI (A.). *Ricreazioni geometrico-fisicali policrome.* Milano, Val-lardi, 1910. 4to.

AUSSANT-CARA (P.). *Sulla discussione dei problemi riducibili al secondo grado, con varie applicazioni.* Livorno, Giusti, 1910. 8vo. 52 pp. L. 1.30

CLARIN (J.). *Cours de mathématiques générales. Tome I: Algèbre, géométrie analytique, calcul différentiel.* Lille, Janny. 4to. 516 pp.

COLIN (C.) et GIROD (J.). *Cours de géométrie, conforme aux programmes officiels du 26 juillet 1909. 2^e année, renfermant les compléments pour toutes les sections.* Paris, Alcan, 1910. 18mo. 6+240 pp. Fr. 2.50

ENRIQUES (F.) e AMALDI (U.). *Nozioni di geometria, ad uso dei ginnasi inferiori.* Bologna, Zanichelli, 1910. 16mo. 158 pp. L. 1.50

FÄRBER (C.). *Grundlehren der Mathematik für Studierende und Lehrer. 1ter Teil, 1ter Band: Arithmetik.* Leipzig, Teubner, 1911. 8vo. 15+410 pp. M. 9.00

FOUBERT (E.). See ROLLET (P.).

FRATTINI (G.). *Geometria intuitiva, per uso delle scuole complementari e del ginnasio inferiore. 3^a edizione.* Torino, Paravia, 1910. 16mo. 117 pp. L. 1.00

GAZZANIGA (P.). See VERONESE (G.).

GIBSON (G. A.) and PINKERTON (P.). *Elements of analytical geometry.* London, Macmillan. 8vo. 498 pp. 7s 6d.

GIROD (J.). See COLIN (C.).

HESSE (A.). *Trigonometrie für Maschinenbauer und Elektrotechniker.* Berlin, Springer, 1911. 8vo. 7+128 pp. Cloth. M. 2.80

KAMBLY und LANGGUTH. *Arithmetik und Algebra, neu bearbeitet von A. Thaer. Ausgabe C: Für Realschulen.* Breslau, Hirt, 1911. 8vo. 96 pp. M. 1.25

LEPOIVRE (G.) et POIRSON (A.). *Cours de géométrie théorique et pratique à l'usage des élèves des écoles pratiques d'industrie, des écoles professionnelles et des écoles primaires supérieures. 2^e édition, revue et augmentée.* Lille, Janny, 1910. 8vo. 6+213 pp.

MACDOUGALL (W.). *Common sense arithmetic.* London, Collins. 8vo. 112 pp. 1s.

NISSSEN (F. J.). *A description of the details and results by the use of the Nissen number system.* Port Clinton, Nissen, 1911. 8vo. 97 pp. \$5.00

OTTO (F.), PETRI (W.) und ZIEGLER (I.). *Mathematik für Lyceen. In 3 Teilen. 1ter Teil: Lehrstoff der Klassen I und II.* Leipzig, Hirt, 1910. 8vo. 7+299 pp. Cloth. M. 4.00

PAPE (L. A. J.). *Courte étude d'instruments et de procédés géométriques (personnels).* Paris, Pape, 1910. 8vo. 4pp.

PAPILLIER (G.). Précis d'algèbre, d'analyse et de trigonométrie, à l'usage des élèves de mathématiques spéciales. 3^e édition. Paris, Vuibert et Nony, 8vo. 452 pp.

PETRI (W.). See OTTO (F.).

PINKERTON (P.). See GIBSON (G. A.).

POIRSON (A.). See LEPOIVRE (G.).

REALE (E.) e RICOZZI (F.). Trattato di disegno geometrico lineare diviso in tre parti, per le scuole medie. Seconda edizione, riveduta ed ampliata. Parte I. Napoli, Casella, 1910. 8vo. 68 pp. L. 1.25

RICOZZI (F.). See REALE (E.).

ROLLET (P.) et FOUBERT (E.). Cours d'algèbre conforme aux programmes des 28 juillet et 28 août 1909. 8^e édition, entièrement refondue. Paris, Alcan, 1910. 18mo. 8+444 pp. Fr. 3.00

ROTHROCK (D. A.). Elements of plane and spherical trigonometry, without tables. New York, Macmillan, 1911. 8vo. 11+147 pp. \$1.10

——. Logarithmic, trigonometric and other tables. New York, Macmillan, 1911. 8vo. 14+99 pp. \$0.60

RUTLAND (N.). Practische Anleitung zum gründlichen Unterrichte in der Buchstabenrechnung. 1ster Teil: Die allgemeine Arithmetik und Algebra. 8te verbesserte Auflage. Bonn, Cohen, 1911. 8vo. 12+440 pp. M. 6.00

SANDERS (H. J.). Elementary graphs, algebra. London, Collins. 8vo. 176 pp. 8d.

——. Elementary graphs, arithmetic. London, Collins. 8vo. 88 pp. 8d.

——. Intermediate arithmetic and geometry. London, Collins. 8vo. 176 pp. 1s. 6d.

SCOTO (G.). Nozioni pratiche di geometria: libro di testo per le tre classi del ginnasio inferiore. 4^a edizione. Palermo, Sandron, 1910. 16mo. 182 pp. L. 2.00

SERRET (J. A.). Trattato di trigonometria. Versione autorizzata dall'autore, con aggiunte per L. Fenoglio. (Collezione Paravia.) Terza edizione. Torino, Paravia, 1911. 8vo. 218 pp. L. 3.00

TESTI (G. M.). Corso di matematiche, ad uso delle scuole secondarie superiori e più specialmente degli istituti tecnici. Vol. II: Algebra elementare. Quinta edizione, nuovamente modificata e corretta. Livorno, Giusti, 1911. 8vo. 13+395 pp. L. 2.80

TRAVNICEK. Geometrie der Ebene. Wien, Deuticke, 1910. M. 1.55

VERONESE (G.). Elementi di geometria, ad uso delle scuole normali, trattati con la collaborazione di P. Gazzaniga. Padova, Drucker, 1911. 8vo. 240 pp. L. 2.50

WATSON (F.). Mathematics for supplementary and continuation classes. London, Oliver and Boyd. 8vo. 238 pp. 2s.

WEBER (K.). Lehrbuch der Stereometrie. Wolfenbüttel, Zwißler, 1911. 8vo. 59 pp. M. 0.75

——. Lehrbuch der Trigonometrie. Wolfenbüttel, Zwißler, 1910. 8vo. 39 pp. M. 0.75

ZIEGLER (I.). See OTTO (F.).

III. APPLIED MATHEMATICS.

- BARR (J. H.) and WOOD (E. H.). *Kinematics of machinery; a brief treatise on constrained motions of machine elements.* 2nd edition. New York, Wiley, 1911. 8vo. 7+264 pp. \$2.50
- BRICARD (R.). *Géométrie descriptive.* (Encyclopédie scientifique. Bibliothèque de mathématiques appliquées.) Paris, Doin, 1911. 18mo. 12+269 pp.
- BRYAN (C. W.). See JOHNSON (J. B.).
- CATHCART (W. L.) and CHAFFEE (J. I.). *The elements of graphic statics and of general graphic methods.* New York, Van Nostrand, 1910. 8vo. 8+312 pp. \$3.00
- CAUBET (P.). *Etude des principales inégalités du mouvement de la lune qui dépendent de l'inclinaison.* (Thèse.) Paris, Privat, 1910. 4to. 98 pp.
- CHAFFEE (J. I.). See CATHCART (W. L.).
- FRIGERI (E.). *Corso di costruzione navale, ad uso degli studenti d'ingegneria navale, dei costruttori e dei naviganti.* (Biblioteca tecnica.) Milano, Hoepli, 1911. 8vo. 18+669 pp. L. 14.00
- GHÉRARDT (M.). *Le gain mathématique à la bourse. La spéculation de bourse considérée comme jeu de pur hasard. Théorie mathématique de la probabilité en matière de cours.* Paris, Amat. 8vo. 182 pp. Fr. 3.00
- HOWE (G. W. O.). See MATHER (T.).
- JAMESON (J. M.). *Elementary practical mechanics.* 2nd edition. New York, Longmans, 1911. 12mo. 12+321 pp. \$1.60
- JOHNSON (J. B.), BRYAN (C. W.) and TURNHAURE (F. E.). *Theory and practice of modern framed structures.* 9th edition, rewritten. Part 2: *Statically indeterminate structures and secondary stresses.* New York, Wiley, 1911. 8vo. 16+538 pp. \$4.00
- JÜPTNER (H. von). *Das chemische Gleichgewicht auf Grund mechanischer Vorstellungen.* Leipzig, Teubner, 1910. 8vo. 5+367 pp. Cloth. M 12.50
- KONDEREF (I. P.). *L'aplanétisme des surfaces et des lentilles elliptiques et hyperboliques.* Genève, Atar. 8vo. 78 pp.
- MASCART (M.). *Eléments de mécanique rédigés conformément au programme de l'enseignement scientifique dans les lycées.* 9e édition. Paris, Hachette, 1910. 8vo. 200 pp. Fr. 3.00
- MATHER (T.) and HOWE (G. W. O.). *Exercises in electrical engineering for the use of second year students in universities and technical colleges.* New York, Longmans, 1911. 12mo. 4+71 pp. \$0.50
- MAYOR (B.). *Statique graphique des systèmes de l'espace.* Paris, Gauthier-Villars. 8vo. 4+208 pp. Fr. 8.00
- MERRIMAN (M.). *The American civil engineers' pocket-book.* New York, Wiley, 1911. 16mo. 8+1380 pp. Morocco. \$5.00
- MÜLLER. *Technische Übungsaufgaben für darstellende Geometrie.* 3tes Heft. Wien, Deuticke, 1910. M. 1.25
- ORTU CARBONI (S.). *Elementi di geometria descrittiva. Proiezione ortogonale o di Monge.* (Collezione Paravia.) Seconda edizione, riveduta, riordinata ed ampliata. Torino, Paravia, 1910. 8vo. 182 pp. L. 2.80

- SAMES (C. M.). Pocket-book of mechanical engineering. 4th edition. Jersey City, Sames, 1910. Flexible leather. \$2.00
- SCHULTZ (E.). Mathematische und technische Tabellen für Maschinenbauschulen und für den Gebrauch in der Praxis. Ausgabe II A mit Logarithmen. 8te Auflage. Essen, Baedeker, 1911. 8vo. 10+311 pp. M. 2.80
- SOMMERFELDT (E.). Die Kristallgruppen nebst ihren Beziehungen zu den Raumgittern. Dresden, Steinkopff, 1911. 7+79 pp. M. 3.00
- SPANGLER (H. W.). Notes on thermodynamics. Part 1. New York, Wiley, 1911. 12mo. 7+80 pp. \$1.00
- STEINMETZ (C. P.). Engineering mathematics; a series of lectures delivered at Union College. New York, McGraw-Hill, 1911. 4to. 17+292 pp. \$3.00
- TURNHAURE (F. E.). See JOHNSON (J. B.).
- WITTENBAUER (F.). Aufgaben aus der technischen Mechanik. 3te (Schluss-) Band: Flüssigkeiten und Gase. Berlin, Springer, 1911. 8vo. 8+328 pp. Cloth. M. 6.80
- WOOD (E. H.). See BARR (J. H.).

FEBRUARY MEETING OF THE AMERICAN
MATHEMATICAL SOCIETY.

THE one hundred and fifty-second meeting of the Society was held in New York City on Saturday, February 25, 1910. The following thirty-eight members attended the two sessions :

Professor W. J. Berry, Professor G. D. Birkhoff, Professor Joseph Bowden, Professor E. W. Brown, Mr. R. D. Carmichael, Professor F. N. Cole, Professor J. L. Coolidge, Dr. H. B. Curtis, Professor L. P. Eisenhart, Professor H. B. Fine, Professor W. B. Fite, Dr. D. C. Gillespie, Professor C. C. Grove, Professor H. E. Hawkes, Mr. Robert Henderson, Professor Percy Hodge, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Mr. W. C. Krathwohl, Dr. N. J. Lennes, Mr. P. H. Linehan, Professor James Maclay, Mr. A. R. Maxson, Dr. E. J. Miles, Dr. H. W. Reddick, Professor L. W. Reid, Professor R. G. D. Richardson, Mr. L. P. Sicheloff, Professor D. E. Smith, Professor P. F. Smith, Mr. W. M. Smith, Professor Elijah Swift, Professor H. D. Thompson, Dr. M. O. Tripp, Professor Oswald Veblen, Professor H. S. White, Miss E. C. Williams.

The President of the Society, Professor H. B. Fine, occupied the chair. The Council announced the election of the following persons to membership in the Society : Dr. Elizabeth R. Bennett, University of Nebraska ; Mr. Daniel Buchanan, University of Chicago ; Dr. H. B. Curtis, Columbia University ; Mr. L. L. Dines, University of Chicago ; Professor C. R. MacInnes, Princeton University ; Professor Eva S. Maglott, Ohio Northern University ; Mr. R. E. Root, University of Chicago ; Professor Sarah E. Smith, Mount Holyoke College. Six applications for membership in the Society were received.

The following papers were presented at this meeting :

(1) Dr. E. J. MILES : "Some properties of space curves minimizing a definite integral with discontinuous integrand."

(2) Dr. N. J. LENNES : "A necessary and sufficient condition for the uniform convergence of a certain class of infinite series."

(3) Dr. N. J. LENNES : "Duality in projective geometry."

(4) Professor G. A. MILLER : "The number of abelian subgroups in the possible groups of order 2^n ."

(5) Professor C. N. MOORE: "On the uniform convergence of the developments in Bessel functions."

(6) Professor G. D. BIRKHOFF: "A direct method for the summation of developments in Lamé's functions and of allied developments."

(7) Professor EDWARD KASNER: "Equitangentials in space."

(8) Professor EDWARD KASNER: "Conformal and equiangular invariants of horn angles."

(9) Professor J. A. EIESLAND: "On a contact transformation in physics."

(10) Dr. D. C. GILLESPIE: "Definite integrals containing a parameter."

(11) Professor JOSEPH BOWDEN: "The Russian peasant method of multiplication."

(12) Dr. N. J. LENNES: "A direct proof of the theorem that the number of terms in the expansion of an infinite determinant is of the same potency as the continuum."

(13) Professor HARRIS HANCOCK: "On algebraic equations that are connected with the cyclotomic equations and the realms of rationality which they determine" (preliminary communication).

(14) Professor W. B. FITE: "Irreducible homogeneous linear groups of order p^n and of degree p or p^2 ."

In the absence of the authors, the papers of Professors Miller, Moore, Eiesland and Hancock were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. In dealing with integrals of the form

$$J = \int F(x, y, z, x', y', z') dt$$

it is customary to assume that F is a continuous function of its six arguments. In Dr. Miles's paper some properties of a minimizing curve

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

are studied when the integrand has a finite discontinuity for points (x, y, z) on a given surface. Three of the necessary conditions which such a curve must satisfy in order to minimize the integral are derived from the known conditions in the ordinary space problem of the calculus of variations. A fourth con-

dition, corresponding to the Jacobi condition, is then derived and it is shown that these conditions, with slight alterations, are also sufficient.

2. In this paper Dr. Lennes gives a necessary and sufficient condition for the uniform convergence over a certain interval of the series

$$\sum_{n=0}^{\infty} U_n(x) = f(x)$$

provided each term of the series and also $f(x)$ are continuous functions in the given interval.

3. In this paper Dr. Lennes gives two sets of fundamental propositions, each sufficient to characterize general projective space. The first set is stated in terms of the abstract symbols "point" and "plane" and two undefined relations "point on plane" and "plane on point." The second set is stated in terms of point, line, and plane, and six undefined relations point on line, line on point, etc. Each set of propositions is dual with respect to point and plane.

4. The theorem that every group G of order p^m , p being an arbitrary prime number, contains an abelian subgroup of order p^a whenever $m > \frac{1}{2}\alpha(\alpha - 1)$ was proved in the *Messenger of Mathematics*, volume 27 (1897-98), page 120. In a later number of the same journal, volume 36 (1906-07), page 79, it was observed that the given theorem can be stated more completely by adding that G contains an *invariant* abelian subgroup of order p^a whenever $m > \frac{1}{2}\alpha(\alpha - 1)$. Moreover, it was observed in this later article that in the very special case where $p = 2$ and $\alpha = 4$ it is possible to extend the theorem, since every group of order 64 contains an abelian invariant subgroup of order 16. In the present paper Professor Miller proves that the theorem in question can be extended for all values of $\alpha > 3$ when $p = 2$. The following theorems are established:

Every group of order 2^m contains an invariant abelian subgroup of order 2^β whenever $m \geq \frac{1}{2}\beta(\beta - 1)$. The number of the abelian subgroups of order p^a in any group of order p^m is of the form $kp + 1$ whenever it is not zero.

5. This paper supplements an earlier one* presented to the Society by Professor Moore. It adds to the results there

* Abstract in BULLETIN, vol. 16 (1909-10), p. 173.

obtained for developments in terms of Bessel functions of order zero, and extends the discussion to Bessel functions of order ν , where $\nu > 0$. The combined papers have been offered for publication to the *Transactions*.

6. A pair of ordinary linear differential equations of the second order

$$L(u) \equiv \frac{d^2 u}{dx^2} + \phi(x)u + [\lambda a(x) + \mu b(x)]u = 0,$$

$$M(v) \equiv \frac{d^2 v}{dy^2} + \psi(y)v + [\lambda c(y) + \mu d(y)]v = 0,$$

$$[a(x)d(y) - b(x)c(y) > 0],$$

will both have solutions $u_i(x)$, $v_i(y)$ which vanish for $x = m_1, m_2$, $y = n_1, n_2$ respectively for certain sets λ_i, μ_i of values of the parameters λ, μ . These solutions give rise to the formal development of an arbitrary function $f(x, y)$ of two variables,

$$f(x, y) \sim \sum c_i u_i(x) v_i(y),$$

including as a special case developments in Lamé's products. Professor Birkhoff treats the question whether this series represents the function by the use of a contour integral

$$\int_C \int_D \int_{m_1}^{m_2} \int_{n_1}^{n_2} f(\xi, \eta) G(x, \xi) H(y, \eta) d\xi d\eta d\lambda d\mu,$$

where G and H are the one-dimensional Green's functions belonging to the first and second of the given equations respectively, under the given boundary conditions, and C and D are properly chosen contours in the λ and μ planes respectively. This integral yields an explicit expression for the sum of any number of terms of the given series, and, by means of a study of the asymptotic character of the solution of the given equations for λ and μ large in absolute value, this series is shown to tend toward $f(x, y)$, where $f(x, y)$ is continuous, under suitable restrictions. The limiting value at a point of discontinuity is also discussed.

The method outlined above is of decidedly greater power than that of integral equations which Hilbert has recently employed, and is to be characterized as an extension of the method of residues.

7. Given any congruence of curves in space, we may obtain ∞^1 equitangential congruences by a construction analogous to that employed by Scheffers in the case of plane systems. Professor Kasner studies the ∞^8 curves thus generated. The simplest result is that for the ∞^1 curves having a common tangent line there is a homographic correspondence between the point of contact and the osculating plane. The theory of isogonals, it may be remarked, differs radically from the plane case since the number of such trajectories for a space congruence is ∞^2 .

8. In the general case of horn angles formed by two curves having contact of the k th order, Professor Kasner shows that there exists a unique conformal invariant I_k and a unique equi-long invariant J_k . The necessary and sufficient condition for equivalence in each group is that the two angles have the same order of contact and equal invariants.

In the simplest case, contact of the first order, the invariants are

$$I = \left(\frac{d\gamma_1}{ds_1} - \frac{d\gamma_2}{ds_2} \right) / (\gamma_1 - \gamma_2)^2, \quad J = \left(\frac{dr_1}{d\theta_1} - \frac{dr_2}{d\theta_2} \right) / (r_1 - r_2)^2,$$

where θ denotes inclination, s arc, γ curvature, and r radius of curvature, the subscripts 1 and 2 referring to the two curves forming the angle. If the values of γ and r in terms of θ and s are inserted, the two expressions differ only by the interchange of the letters θ and s .

All contact transformations turn horn angles into horn angles of the same order of contact. Those which preserve the value of I (or J) constitute the conformal (or equi-long) group.

In a previous paper the author showed that ordinary curvilinear angles have under the conformal group no invariant beyond the magnitude of the angle, except when that magnitude is commensurable to 2π . Under the equi-long group, the figure formed by two curves with a common tangent has the length of that tangent for its sole invariant.

9. Professor Eiesland's paper is in abstract as follows: A contact transformation

$$\delta x = \frac{\partial w}{\partial p} \delta t, \quad \delta y = \frac{\partial w}{\partial q} \delta t, \quad \delta z = \left(p \frac{\partial w}{\partial p} + q \frac{\partial w}{\partial q} - w \right) \delta t, \\ \delta p = 0, \quad \delta q = 0$$

whose characteristic function w is a function of p and q only will represent a wave motion in a non-isotropic body, the velocity depending only on the direction of the motion. A surface element is transformed parallel to itself, while a point is transformed into a surface

$$F\left(\frac{x_1-x}{t}, \frac{y_1-y}{t}, \frac{z_1-z}{t}\right) = 0.$$

The velocity v , normal to the wave, is expressed by the relation

$$v = \frac{w}{\sqrt{1+p^2+q^2}}.$$

In the case of wave motion in a biaxial crystal v is defined by the equation

$$(1+p^2+q^2)v^4 - [(b^2+c^2)p^2 + (a^2+c^2)q^2 + (a^2+b^2)]v^2 + b^2c^2p^2 + a^2c^2q^2 + a^2b^2 = 0,$$

that is, w is defined by the equation

$$w^4 - [(b^2+c^2)p^2 + (a^2+c^2)q^2 + a^2+b^2]w^2 + (b^2c^2p^2 + a^2c^2q^2 + a^2b^2)(1+p^2+q^2) = 0.$$

The corresponding contact transformation may now be set up. It is found that it is of the form

$$x_1-x = \frac{vp}{\sqrt{1+p^2+q^2}} \left[\frac{a^2-r^2}{a^2-v^2} \right] t, \quad y_1-y = \frac{vq}{\sqrt{1+p^2+q^2}} \left[\frac{b^2-r^2}{b^2-v^2} \right] t,$$

$$z_1-z = \frac{-v}{\sqrt{1+p^2+q^2}} \left[\frac{c^2-r^2}{c^2-v^2} \right] t, \quad p_1=p, \quad q_1=q,$$

where

$$r^2 = \sum \frac{(x_1-x)^2}{t^2}.$$

Elimination of p and q from these equations gives the wave surface in the most general form, viz.:

$$\frac{(x_1-x)^2}{t^2(r^2-a^2)} + \frac{(y_1-y)^2}{t^2(r^2-b^2)} + \frac{(z_1-z)^2}{t^2(r^2-c^2)} = 1.$$

For $x = y = z = 0$ and $t = 1$ (t being interpreted as time) we have the wave surface as usually given.

The advantage of the above form consists in this that it puts directly in evidence Huygens' principle.

The wave motion in a biaxial crystal may be represented mathematically by a contact transformation whose characteristic function is

$$w = \sqrt{1 + p^2 + q^2} v,$$

v^2 being a root of the quadratic equation above.

10. In this paper Dr. Gillespie considers the definite integral

$$\int_a^b f(\alpha, x) dx,$$

where $f(\alpha, x)$ is an integrable * function of x for all values of α in a closed interval. The question to which an answer is sought is: what conditions must be imposed upon the $f(\alpha, x)$ so that

$$\left| \int_a^b f(\alpha, x) dx - \sum f(\alpha, x_i)(x_i - x_{i-1}) \right| < \epsilon \quad (i = 1, \dots, n),$$

where $|x_i - x_{i-1}| < \eta$ for all values of α ? η and ϵ have the usual significance, and the fixed law of subdivision is such that the limit of $x_i - x_{i-1}$ as n becomes infinite is zero. Some examples are given showing the necessity of imposing conditions on $f(\alpha, x)$; sufficient conditions are then derived. Supposing $f(\alpha, x)$ to be a continuous function of α for all values of x , the necessary and sufficient condition that

$$\int_a^b f(\alpha, x) dx$$

be a continuous function of α is obtained. A further application of this theory is found in certain problems in integral equations.

11. In this paper Professor Bowden gives a proof of the correctness of the Russian peasant method of multiplication, for the performance of which it is necessary to know only how to add, how to double a number, and how to divide by two,

* Riemann's definition of the definite integral.

obtaining the exact or lower approximate quotient. The method may be described as follows:

To multiply a by b write down $a \times b$. Under a write the exact or lower quotient obtained by dividing it by 2; under this quotient write the quotient obtained by dividing it by 2; and so on, until we obtain the quotient 1.

Under b write its double, under this double its double, and so on, until we have as many numbers in the second column as in the first.

Next add the numbers in the second column which correspond to odd numbers in the first. The result is the product of a and b .

The proof depends on the theorems given on pages 167, 168 in Part I of Chrystal's Text-Book of Algebra.

12. The theorem in question is proved by Dr. Lennes by setting the terms of the expansion of the determinant into one-to-one correspondence with the points of a line segment.

13. Professor Hancock's paper is in abstract as follows: If the three trinomials $x^3 + a_1x + 1$, $x^3 + a_2x + 1$, $x^3 + a_3x + 1$ are multiplied together and the coefficients of the resulting expression equated to those of $(x^7 - 1)/(x - 1)$, it is seen that $a_1 = -2 \cos 2\pi/7$, $a_2 = -2 \cos 4\pi/7$, $a_3 = -2 \cos 6\pi/7$ are the roots of the equation $f_3(x) = x^3 - x^2 - 2x + 1 = 0$. In a similar manner are derived $f_4(x) = x^4 - x^3 - 3x^2 + 2x + 1 = 0$ (roots: $x = -2 \cos 2\nu\pi/9$ ($\nu = 1, 2, 3, 4$)), $f_5(x) = x^5 - x^4 - 4x^3 + 3x^2 + 3x - 1 = 0$ (roots: $x = -2 \cos 2\nu\pi/11$ ($\nu = 1, 2, 3, 4, 5$)), ...

The following relations are at once evident:

$$f_k(2) = 1 \quad (k = 1, 2, 3, \dots, n),$$

$$f_k(0) = (-1)^{\frac{\tau^2-1}{8}} = \prod_{\nu=1}^{\nu=k} 2 \cos \frac{2\nu m\pi}{\tau} \quad \left(\begin{array}{l} m \text{ an odd integer} \\ \tau = 2k + 1 \end{array} \right).$$

The latter formula is of importance in the theory of quadratic residues.

Write

$$\vartheta_\nu = -2 \cos \frac{2\nu\pi}{2k+1},$$

where ν is any one of the integers 1, 2, ..., k and where at

first $2k + 1$ is taken as a prime integer. The latter restriction is eventually removed.

Let $\Omega_k = R(\vartheta_k)$ be the realm of rationality determined by ϑ_k , where, as seen above, ϑ_k is a root of $f_k(x) = 0$. Several different ways are given for the determination of the discriminant of this realm. N denoting the norm of the quantity which follows it, it is found that

$$\Delta(1, \vartheta_k, \vartheta_k^2, \dots, \vartheta_k^{k-1}) = (-1)^{\frac{k(k-1)}{2}} N\{f'_k(\vartheta_k)\} = (2k + 1)^{k-1}.$$

It is also seen that this number is the basal invariant [Grundzahl = $\Delta(\Omega_k)$] of the realm Ω_k , the index of the above basis being unity. The quantities ϑ_k^i are algebraic units in Ω_k ; and, if $\mu = 2 + \vartheta_k$, then $1, \mu, \mu^2, \dots, \mu^{k-1}$ form a basis of all algebraic integers in Ω_k .

Similar results are derived for the equations, whose roots are

$$\sigma_\nu = 2 \sin \frac{2\nu\pi}{2k+1} \quad (\nu = 1, 2, 3, \dots, 2k; k = 1, 2, \dots, n),$$

and the results which have hitherto been derived are compared with those which follow from the equations

$$(x^{2k+1} - 1)/(x - 1) = 0 \quad (k = 1, 2, \dots, n).$$

14. A group of order p^n (p a prime) and class one, two, or three, can be simply isomorphic with irreducible homogeneous linear groups of only one degree. This is also true of those groups all of whose non-invariant commutators give invariant commutators besides identity. Moreover no group of order p^n can be simply isomorphic with irreducible groups of just two different degrees. Professor Fite considers the question, suggested by the consideration of these facts, as to whether any group of order p^n can be simply isomorphic with irreducible groups of different degrees. The question is not answered, but a few facts having a bearing thereon are established.

F. N. COLE,
Secretary.

ON THE CLASSIFICATION OF CRYSTALS.

BY PROFESSOR PAUL SAUREL.

(Read before the American Mathematical Society, December 29, 1910.)

THE determination of the various types of crystal symmetry is a problem which has attracted the attention of many physicists and mathematicians. The first solutions of the problem are those of Hessel,* of Bravais,† and of Gadolin,‡ while the latest and most elegant is that of Lorentz.§ In what follows I should like to show that by combining theorems due to Curie and to Lorentz it is very easy to enumerate and to remember the different types of symmetry.

A few preliminaries are necessary. In the first place, if the line joining two points P and P' is bisected by a point O , we shall say that P' is the inverse of P with respect to O . In the next place we shall use the phrase inversion with respect to a point to denote the operation which consists in replacing every point of a figure by its inverse with respect to the given point.

Let us consider any system of points. If the system is such that rotation through a suitable angle about a line OA transforms the system of points into itself, we shall say that OA is an axis of direct symmetry. It is obvious that if rotation through an angle α transforms the system into itself, rotation through any multiple of α will also transform the system into itself. It follows that the number of different rotations which transform the system into itself will be infinite unless α is commensurable with 2π . Accordingly, if we limit ourselves to systems which admit of a finite number only of transformations, α must be equal to $2\pi m/q$, where m and q are integers prime to each other.

* J. F. C. Hessel, *Krystallometrie oder Krystallonomie und Krystallographie*; Gehler's *Physikalisches Wörterbuch*, vol. 5, 1830, p. 1062; Ostwald's *Klassiker der exakten Wissenschaften*, no. 88.

† A. Bravais, "Mémoire sur les polyèdres de forme symétrique," *Journal de mathématiques pures et appliquées*, vol. 14 (1849), p. 141; Ostwald's *Klassiker der exakten Wissenschaften*, no. 17.

‡ A. Gadolin, "Mémoire sur la déduction d'un seul principe de tous les systèmes crystallographiques," *Acta Societatis Scientiarum Fennicae*, Helsingfors, vol. 9 (1871), p. 1; Ostwald's *Klassiker der exakten Wissenschaften*, no. 75.

§ H. A. Lorentz, "Über die Symmetrie der Kristalle", *Abhandlungen über theoretische Physik*, vol. 1 (1907), p. 299.

Moreover, it can be shown without difficulty that the different rotations about OA correspond to the first $q - 1$ multiples of the angle $2\pi/q$. The axis OA is said to be of order q , and the rotations about it may be performed in either direction. It is clear that if OA is an axis of direct symmetry of order q , the same is true of OA' , the inverse of OA with respect to O .

The system of points may be such that it is possible to draw through a point O two sets of rectangular coordinate axes, one right-handed and the other left-handed, in such a way that to every point of the system whose coordinates with respect to one set of axes are the numbers x, y, z , there corresponds a point of the system whose coordinates with respect to the other set of axes are the same numbers x, y, z . When this is the case, it is easy to see that it is possible to transform the system into itself by means of a rotation followed by an inversion. Indeed, it is clear that a rotation can be found which will bring one set of coordinate axes into coincidence with the inverses of the other set; this rotation, followed by an inversion with respect to O , will obviously transform every point of the system into a point of the system. If the rotation is unnecessary, so that a mere inversion with respect to O transforms the system into itself, we shall say that O is a center of symmetry. In the general case, where a rotation is necessary, we shall call the axis of rotation an axis of inverse symmetry.

If, as before, we limit ourselves to systems which admit of a finite number only of transformations, it follows that the angle of rotation associated with an axis of inverse symmetry must be equal to $2\pi m/q$, where m and q are integers prime to each other. If q is odd, it can be shown that the axis of inverse symmetry is equivalent to an axis of direct symmetry of order q and a center of symmetry; if, on the contrary, q is even, the various operations connected with the axis of inverse symmetry are equivalent to the repetition of a rotation through an angle $2\pi/q$ accompanied by an inversion.

To establish this, we observe that the rotation about the axis of inverse symmetry may precede or follow the inversion; the result in each case is the same. Moreover, two successive inversions always transform a system into itself. Accordingly, q rotations through an angle $2\pi m/q$, accompanied by q inversions, are equivalent to rotation through an angle $2\pi m$ followed by q inversions; if q is odd, the operation is thus equivalent to a single inversion and the system possesses a center of sym-

metry. From this it follows immediately that a mere rotation through the angle $2\pi m/q$ transforms the system into itself. Accordingly, an axis of inverse symmetry of odd order is equivalent to an axis of direct symmetry of the same order and a center of symmetry.

Again, the repetition k times of the operation of rotation through an angle $2\pi m/q$ and inversion is equivalent to a single rotation of angle $2\pi mk/q - 2\pi l$, where l is an integer, followed by k inversions. Since

$$\frac{2\pi mk}{q} - 2\pi l = \frac{2\pi}{q}(mk - ql),$$

and since we can always choose k and l so that

$$mk - ql = \pm 1,$$

it follows that one of the operations connected with the axis of inverse symmetry consists in a rotation of angle $2\pi/q$ followed by k inversions. From the last equation it follows that k must be prime to q ; accordingly, when q is even, k must be odd. Thus, when q is even, all the operations connected with the axis of inverse symmetry are repetitions of a rotation through an angle $2\pi/q$ accompanied by an inversion. There are $q - 1$ different operations and no one of them is an inversion.

It is now easy to see that the crystallographic problem of the determination of the various possible types of crystal symmetry is equivalent to the problem of determining the various sets of axes of direct and of inverse symmetry passing through a common point, which a system can possess; the point of intersection of the axes may or may not be a center of symmetry.

For this purpose it is sufficient to recall that crystals possess two kinds of axes of symmetry. An axis of symmetry of the first kind is characterized by the fact that the physical properties of the crystal with respect to any fixed directions in space remain unchanged when the crystal is turned through a suitable angle about the axis. Accordingly, if we consider a portion of the crystal bounded by a spherical surface and if we draw the axis of symmetry through the center of this sphere, we may say that a rotation about the axis transforms the crystal into itself.

An axis of symmetry of the second kind is characterized by the fact that if the crystal is turned through a suitable angle about such an axis, its physical properties in its new position

and with respect to any fixed directions in space are identical with the properties of the crystal in its first position and with respect to the fixed directions which are the inverse of those just mentioned. Accordingly, if, as before, we consider a portion of the crystal bounded by a spherical surface, and if, through the center, we draw an axis of symmetry of the second kind, we may say that a rotation about this axis, followed by an inversion with respect to the center of the sphere, transforms the crystal into itself.

We shall now proceed to the enumeration of the different sets of axes of direct and of inverse symmetry, passing through a common point, that a system can possess. From what precedes, it is obvious that this problem is identical with the problem of finding all the finite groups of operations that can be generated by rotations about a point and inversion with respect to it.

Let us begin by finding the groups that consist of rotations only; it will then be easy to find the groups that contain inversions also. In the first place, it is clear that, if OA be an axis of direct symmetry of order q , the different positions which OA takes when the system is subjected to all the rotations of the group are also axes of direct symmetry of order q . Every axis thus belongs to a definite set of equivalent axes and the various axes of the system can be grouped into one or more sets of equivalent axes. In enumerating the axes of direct symmetry we must distinguish between OA and its inverse OA' ; OA and OA' will belong to the same or to different sets of equivalent axes according as it is or is not possible to bring OA into coincidence with OA' .

We can now establish the following two theorems which are due to Curie* and which suffice to determine the possible groups of rotations.

THEOREM I. *If n be the number of different rotations, including the identical rotation, which transform a system of points into itself, and if p be the number of different positions which an axis of order q assumes when the system is subjected to these n rotations, then*

$$(1) \quad pq = n.$$

* P. Curie, "Sur les questions d'ordre," *Bulletin de la Société minéralogique de France*, vol. 8 (1884), p. 89; *Oeuvres*, p. 70. "Sur les répétitions et la symétrie," *Comptes rendus de l'Académie des Sciences*, vol. 100 (1885), p. 393; *Oeuvres*, p. 114.

THEOREM II. *If a system possesses a set of p equivalent axes of order q , a set of p' equivalent axes of order q' , a set of p'' equivalent axes of order q'' , ..., and if k be the number of these sets, then*

$$(2) \quad p(q-1) + p'(q'-1) + p''(q''-1) + \dots = 2(n-1),$$

or

$$(3) \quad p + p' + p'' + \dots = (k-2)n + 2,$$

or finally

$$(4) \quad \frac{1}{q} + \frac{1}{q'} + \frac{1}{q''} + \dots = k - 2 + \frac{2}{n}.$$

To establish equation (1) it is sufficient to observe that for each of the p positions which the axis of rotation under consideration can assume, the system can assume any one of q positions obtained by rotation about this axis; the system can thus be transformed into itself by any one of pq different operations. Moreover, it is obvious that the pq different positions thus obtained are the only ones which the system can take; thus pq is equal to n .

To establish equation (2) we observe that the system can be transformed into itself by any one of $q-1$ rotations about each of the p axes of the first set, by any one of $q'-1$ rotations about each of the p' axes of the second set, by any one of $q''-1$ rotations about each of the p'' axes of the third set, ..., and that in this enumeration every rotation, except the identical rotation, has been counted twice; OA and its inverse OA' may belong to the same or to different sets of axes, in either case the rotations about AA' have been counted twice. Equation (3) follows without difficulty from equations (1) and (2), and in like manner equation (4) follows from equations (1) and (3).

The solutions of equations (1) and (4) will give us all the sets of axes of direct symmetry that can coexist. There is no difficulty in finding these solutions; it is found that when $k=1$ or $k>3$ there are no solutions, and the results for $k=2$ and $k=3$ are given in the following table:

k	p	p'	p''	q	q'	q''	n
2	1	1		n	n		n
3	m	m	2	2	2	m	$2m$
	6	4	4	2	3	3	12
	12	8	6	2	3	4	24
	30	20	12	2	3	5	60

We shall now show that corresponding to each solution in the above table there exists one and but one arrangement of the axes of symmetry in space. When $k = 2$, there are two sets of equivalent axes each consisting of a single axis of order n ; it is obvious that these two axes must be inverses of each other. The geometric configuration corresponding to this solution thus exists and it is unique. The corresponding groups of rotations are called the cyclic groups.

When $k = 3$, let us draw a unit sphere about O as a center and let us mark on the sphere the points where the three sets of axes of symmetry pierce it. It is well known that if we construct any spherical triangle ABC , a rotation about OA through an angle $2A$, followed by a rotation about OB through an angle $2B$, is equivalent to a rotation about OC through an angle $2C$. Accordingly, if OA and OB are axes of symmetry of orders q and q' and if the angles A and B are respectively equal to π/q and π/q' , the line OC will also be an axis of symmetry of the system and the half-angle of rotation associated with it will be equal to C or to some aliquot part of C . We shall establish the following theorem:

THEOREM III. *If OA and OB are axes of symmetry of orders q and q' belonging to different sets of equivalent axes and if the angle between OA and OB is not greater than the angle between any two axes of symmetry belonging to different sets, then, if we construct the spherical triangle ABC with angles A and B respectively equal to π/q and π/q' , the line OC will be an axis of symmetry of order q'' belonging to the third set of equivalent axes and the angle C will be equal to π/q'' .*

In the first place, the half-angle of rotation associated with the axis OC is equal to the angle C and not to an aliquot part

of C . For if an aliquot part of C were the half-angle of rotation associated with the axis OC , we could draw through C an arc making this angle with AC and thus form a triangle ACD ; the line OD , lying between OA and OB , would thus, contrary to hypothesis, be an axis of symmetry.

In the next place, it is obvious that if the angle C differs from the angles A and B , the axis of symmetry OC belongs to the third set of equivalent axes. It remains to prove that this is the case even when the angle C is equal to one of the angles A or B . Accordingly, let us suppose that in the triangle ABC the angles B and C are equal and that OB and OC belong to the same set of equivalent axes. It is easy to see that no axis belonging to the third set of axes can pierce the sphere within the triangle ABC or on either of the sides AB and AC , for any such axis would make with OA a smaller angle than does OB . This becomes evident if we remember that no one of the angles of the triangle is greater than $\frac{1}{2}\pi$ and that consequently the perpendiculars from the vertices to the opposite sides do not fall without the triangle. Further, no axis of the third set can meet the side BC . The reason just given applies here also except when AB and AC are quadrants; but in that case the arc BC cannot be greater than a quadrant since the angle A is not greater than $\frac{1}{2}\pi$, and consequently any axis meeting the side BC would form with OB an angle less than the angle between OB and OA . Thus, no axis belonging to the third set can pierce the sphere within the triangle ABC or on its perimeter.

Let us now construct a triangle BCA_1 equal to BCA and having the side BC in common. The vertex A_1 corresponds to an axis belonging to the same set as OA , for OA_1 is the position which OA takes when it is revolved about OB through an angle $2B$. In like manner, we can construct triangles ACB_1 and ABC_1 equal to ABC ; the vertices B_1 and C_1 will correspond to axes belonging to the same set as OB and OC . By repeating this construction we obtain a network of triangles the vertices of which correspond to axes belonging to the same sets as OA , OB , OC . This network must cover the sphere completely and there can be no overlapping. Indeed, it is obvious that the construction can be continued as long as any portion of the surface is uncovered. On the other hand there can be no overlapping, for, if there were, there would lie within a triangle such as ABC a point D corresponding to an axis belonging to

one of the first two sets. If we remember that the angles A , B , C are not greater than $\frac{1}{2}\pi$ and that consequently the perpendiculars from the vertices to the opposite sides do not fall without the triangle, it is clear that DA and DB are each less than AB . We should then have, contrary to the hypothesis, an axis nearer to OA or to OB than these two are to each other.

The sphere has thus been covered by a network of triangles equal to ABC , the vertices of which correspond to axes of the first and second sets. Moreover, it has been shown that no axis of the third set can pierce the triangle ABC . In like manner it can be shown that no axis of the third set can pierce a triangle belonging to the network of equal triangles. There is thus no place for the axes belonging to the third set. We must therefore abandon the hypothesis that OC is an axis belonging to the second set and our Theorem III is established.

Let us now consider a triangle ABC of the kind described in Theorem III. By means of the construction described above we can obtain a set of triangles equal to or symmetrical with ABC and the vertices of these triangles will correspond to axes belonging to the three sets of axes. Moreover, it can be shown as above that this network of triangles will cover the sphere completely and that there will be no overlapping. We are thus led to the problem of determining the values of q , q' , q'' such that we can cover the sphere with a set of triangles whose angles are π/q , π/q' , π/q'' . Since the area of one of these triangles is

$$\frac{\pi}{q} + \frac{\pi}{q'} + \frac{\pi}{q''} - \pi,$$

we must have, if N denote the number of triangles,

$$N \left(\frac{\pi}{q} + \frac{\pi}{q'} + \frac{\pi}{q''} - \pi \right) = 4\pi,$$

or

$$(5) \quad \frac{1}{q} + \frac{1}{q'} + \frac{1}{q''} = 1 + \frac{4}{N}.$$

As this equation differs from equation 4 for $k=3$ only in the substitution of N for $2n$, its solutions can be at once written down. They are given by the following table

q	q'	q''	N
2	2	m	$4m$
2	3	3	24
2	3	4	48
2	3	5	120

It is obvious that if there exists a network of triangles corresponding to any one of these solutions it is unique; for the triangle ABC is determined when its angles are given and the other triangles are determined without ambiguity when ABC is given. Now it is well known that there exists a set of triangles corresponding to each of the above solutions. The $4m$ triangles corresponding to the first solution are obtained by dividing the circumference of a great circle into $2m$ equal parts and connecting the points of division with the poles of the circle. The radii to the poles of the great circle form a set of 2 equivalent axes of order m , while the $2m$ radii to the points of division of the great circle fall into two sets of m equivalent axes of order 2. The groups of rotations corresponding to this case are called the dihedral groups.

The configuration corresponding to the second solution in the table is found by inscribing in the sphere a pair of polar tetrahedrons. Each vertex of the first tetrahedron lies within a triangle formed by vertices of the other; by connecting each vertex of the first tetrahedron with the vertices and the mid-points of the sides of the surrounding triangle we find 24 triangles whose angles are $\pi/2, \pi/3, \pi/3$. The vertices of the two tetrahedrons correspond to the two sets of 4 equivalent axes of order 3, while the remaining vertices of the triangles correspond to the 6 axes of order 2. The corresponding group of rotations is called the tetrahedral group.

The configuration corresponding to the third solution in the table is found by inscribing in the sphere a cube and its polar octahedron. Each vertex of the octahedron lies within a quadrilateral formed by vertices of the cube; by connecting each vertex of the octahedron with the vertices and the mid-points of the sides of the surrounding quadrilateral we find 48 triangles whose angles are $\pi/2, \pi/3, \pi/4$. The vertices of the octahe-

dron correspond to a set of 6 equivalent axes of order 4, the vertices of the cube correspond to a set of 8 equivalent axes of order 3, while the remaining vertices of the triangles correspond to the 12 axes of order 2. The corresponding group of rotations is called the octahedral group.

Finally, the configuration corresponding to the last solution in the table is found by inscribing in the sphere a regular dodecahedron and its polar icosahedron. Each vertex of the icosahedron lies within a pentagon formed by vertices of the dodecahedron; by connecting each vertex of the icosahedron with the vertices and the mid-points of the surrounding pentagon we find 120 triangles whose angles are $\pi/2, \pi/3, \pi/5$. The vertices of the icosahedron correspond to a set of 12 equivalent axes of order 5, the vertices of the dodecahedron correspond to a set of 20 equivalent axes of order 3, while the remaining vertices of the triangles correspond to the 30 axes of order 2. The corresponding group of rotations is called the icosahedral group.

Thus the only groups of rotations are the identical rotation and the cyclic, the dihedral, the tetrahedral, the octahedral, and the icosahedral groups.

As Lorentz * has shown, we can now obtain without difficulty the groups of operations that contain inversions as well as rotations. In the first place, we may associate an inversion with each of the groups of rotations; the system will then possess in addition to the axes of symmetry a center of symmetry.

In the next place, since an axis of inverse symmetry of odd order is equivalent to an axis of direct symmetry of the same order and a center of symmetry, we may now limit ourselves to the consideration of axes of inverse symmetry of even order. Furthermore, since the inversions and rotations about the axes of inverse symmetry must, like the rotations about the axes of direct symmetry, transform the group of axes of symmetry into itself; it follows that the only admissible configurations for the axes of symmetry are the configurations which we found in studying the groups of rotations. The groups of operations containing operations corresponding to axes of inverse symmetry will therefore be found by replacing one or more sets of axes of direct symmetry by axes of inverse symmetry of the same orders. Finally, if we consider a triangle of the kind

* L. c.

described in Theorem III, it is easy to see that if one of the three axes OA , OB , OC be replaced by an axis of inverse symmetry, one other must also be replaced by such an axis while the third will remain an axis of direct symmetry.

With these remarks in mind it is easy to enumerate the groups of operations which contain operations corresponding to axes of inverse symmetry. In the first place, we obtain from every cyclic group of even order a new group by replacing the axis of direct symmetry by an axis of inverse symmetry of the same order. In the next place, we obtain from every dihedral group a new group by replacing the two sets of axes of order 2 by axes of inverse symmetry of order 2. Moreover, we obtain a new group from every dihedral group of order $2m$, when m is an even number equal to or greater than 4, by replacing the pair of axes of order m and one set of axes of order 2 by an axis of inverse symmetry of order m and axes of inverse symmetry of order 2. Finally, we obtain a new group from the octahedral group by replacing the axes of orders 2 and 4 by axes of inverse symmetry of the same orders. The tetrahedral and icosahedral groups furnish no new groups.

Our results are summarized in the following table, in which the letters C , D , T , O , I are used to recall the cyclic, the dihedral, the tetrahedral, the octahedral, and the icosahedral groups and the subscripts are used to indicate the orders of the corresponding axes of symmetry. A bar over a subscript indicates that the corresponding axes of direct symmetry have been replaced by axes of inverse symmetry.

TYPES OF SYMMETRY CHARACTERIZED BY THE EXISTENCE OF

Axes of Direct Symmetry	Center of Symmetry and Axes of Direct Symmetry.	Axes of Direct and of Inverse Symmetry
No axis	No axis	
C_n	C_n	$C_{\overline{2n}}$
$D_{2, 2, m}$	$D_{2, 2, m}$	$D_{\overline{2, 2, m}}$ $D_{\overline{2, 2, 2(\kappa+2)}} \quad \kappa \geq 0$
$T_{2, 3, 3}$	$T_{2, 3, 3}$	
$O_{2, 3, 4}$	$O_{2, 3, 4}$	$O_{\overline{2, 3, 4}}$
$I_{2, 3, 5}$	$I_{2, 3, 5}$	

Not all of the types of symmetry enumerated in this table are available as types of crystal symmetry, for the law of rational indices limits the acceptable axes of symmetry to those of the orders 2, 3, 4, 6. With this limitation the table furnishes the 32 types of crystal symmetry, 11 from each of the first two columns and 10 from the third.

NEW YORK.

December 4, 1910.

HORNER'S METHOD OF APPROXIMATION ANTICIPATED BY RUFFINI.

BY PROFESSOR FLORIAN CAJORI.

(Read before the Southwestern Section of the American Mathematical
Society, November 26, 1910.)

BEFORE the nineteenth century no simple elementary practical process was known of computing the coefficients of an equation whose roots are less by a given constant than the roots of a given affected numerical equation $f(x) = 0$. Such a process was invented in the early part of the last century and is contained in the so-called "Horner's method of approximation."

The history of Horner's method, as developed in England, has been traced in detail by Augustus De Morgan.* He quotes extensively from original sources and shows that, of English rivals (among whom were Theophilus Holdred, Peter Nicholson, and Henry Atkinson), none except perhaps Nicholson advanced methods of approximation that equalled Horner's, and none was entitled to priority over Horner. It is well-known that the popularization of Horner's process of approximation in England was due to De Morgan and J. R. Young. Except for the efforts of these men, Horner's paper of 1819 in the *Philosophical Transactions* might have been lost sight of and forgotten. De Morgan was an enthusiast on Horner's method. He taught it with great zeal; he made sport of Cambridge tutors who were not familiar with it;† the preparation of his historical tract, alluded to above, was evidently a labor

* *Companion to the [British] Almanac for 1839*, Art. "Notices of the progress of the problem of evolution," pp. 34-52.

† A. De Morgan, *A Budget of Paradoxes*, London, 1872, pp. 292, 375.

of love. It is still the standard reference to early English research on the subject. That Horner had a rival claimant outside of England, De Morgan never suspected.*

It is the purpose of this paper to show that fifteen years before the publication of Horner's first paper† an almost identical method was published in Italy by Paolo Ruffini, whose name is usually associated with the proof of the impossibility of solving algebraically the quintic equation. Ruffini had no J. R. Young and no De Morgan to urge upon the mathematical public the importance of his method of approximation. The method was overlooked, and now, over a century later, begins the weary process of the "restitution of decayed intelligence."

In 1802 the Italian Scientific Society (of Forty) offered a gold medal for the best method of determining the roots of a numerical equation of any degree. Five dissertations were submitted in competition and in 1804 the medal was awarded to Paolo Ruffini. The dissertation was published by the Society in an unwieldy, large quarto monograph of 175 pages, bearing the title:

"Sopra la determinazione delle radici nelle equazioni numeriche di qualunque grado. Memoria del Dottor Paolo Ruffini, pubblico professore di matematica sublime in Modena, uno dei quaranta della società italiana delle scienze ec. Coronata dalla società medesima. In Modena, MDCCCV. Presso la società tipografica. Con Approvazione."

Ruffini wrote on the subject of approximation again in 1807‡ and 1813.§ In these later articles he gave a simpler exposition of the method and explained its application to the extraction of roots of numbers.

While Horner had no knowledge of Ruffini's publications on

* Budget of Paradoxes, p. 375.

† Horner prepared three important papers on his method of approximation: (1) "A new method of solving numerical equations of all orders, by continuous approximation," *Philosophical Transactions* (London), 1819, Part I, pp. 308-335; (2) An article in *Leybourn's Mathematical Repository*, Vol. 5, Part 2, No. 19, London, 1830; Extracts from this are given in J. R. Young's *Theory and Solution of Algebraical Equations*, London, 1843, pp. 474, 475; (3) "On algebraic transformation," *The Mathematician*, Vol. I, London, 1845, p. 109.

‡ Algebra elementare del Dottor Paolo Ruffini, Modena, MDCCCVII, Capo IV, V, and in an Appendix which is not in my copy of this Algebra and seems to have been bound separately and published in 1808.

§ "Di un nuovo metodo generale di estrarre le radici numeriche." Memoria del Signor Cav. Paolo Ruffini. *Memorie di Matematica e di Fisica della Società Italiana delle Scienze*. Tomo XVI, Parte I. Verona, MDCCXIII, pp. 373-429; Tomo XVII, parte contenente del memorie di matematica, pp. 1-15.

the approximation to the roots of equations, Horner's papers of 1819 and 1845 contain nevertheless some interesting parallelisms to Ruffini's publications of 1804, 1807, and 1813.

In their first publications both authors used the differential calculus in expounding their methods; Ruffini used ordinary derivatives, Horner used Arbogast's derivatives.

In later publications both authors gave simplified explanations of their methods, dispensing with the calculus and using ordinary algebra. Ruffini's simplified exposition appeared in 1807 and 1813; Horner's in 1845 as a posthumous article.*

Both authors pointed out that the methods could be used with advantage in the root-extraction of numbers. Ruffini laid greater stress on this point than did Horner; in fact, Ruffini's publications of 1807 and 1813 consider the process from this point of view alone. Neither author extended the method of approximation so as to render it applicable to the determination of complex roots.

A contrast between the two authors lies in the fact that Horner in 1819 erroneously believed that his method was at once applicable to irrational and transcendental equations. He says: "From the unrestricted nature of the notation employed, it is evident that no class of equations, whether finite, irrational, or transcendental, is excluded from our design."† He gives no other argument nor any illustrative solutions of irrational or transcendental equations, in support of his claim. Ruffini, on the other hand, confined his discussion to rational integral algebraic equations.

A second contrast is this, that Horner in his paper of 1830 carefully explains a special procedure in his method, by which two nearly equal real roots can be separated with certainty. Ruffini never considered this refinement of the method.

To explain Ruffini's method of approximation, we quote from pages 22-25 of his memoir of 1804. The equation (*A*), referred

* This article of 1845 had been read before the Royal Society, June 19, 1823, but (as T. S. Davies explains in an introductory note to the printed paper) was refused publication in the *Philosophical Transactions*, even after it had been considerably curtailed. Horner often stated to Davies that much demur was made to the insertion of his first paper in the *Philosophical Transactions*. "The elementary character of the subject," says Davies, "was the professed objection: his recondite mode of treating it, was the professed passport for its admission." Horner's third article was a simplification of the principles of his process. In this article he says of Budan, I am "satisfied with finding that he had not anticipated my discovery."

† *Philosophical Transactions*, 1819, Part I, pp. 317, 318.

to in the quotation, is as follows:

$$(A) \quad Ax^m + Bx^{m-1} + Cx^{m-2} + \text{ec.} + Sx^2 + Tx + V = 0$$

“Supposto $x = p + y$, determinare un metodo spedito trasformare l' Equazione (A) in un' altra con l' incognita

“Dalle proprietà delle trasformate sappiamo, che, ess il primo membro della (A) (n.° 1), la trasformata richi ottiene ponendo p in luogo di x nella

$$Z + \frac{dZ}{dx}y + \frac{ddZ}{2dx^2}y^2 + \frac{d^3Z}{2 \cdot 3dx^3}y^3 + \text{ec.} + Ay^m = 0.$$

Ora pel (n.° 10) nella ipotesi di $x = p$ abbiamo $Z = 1$ quindi

$$\frac{dZ}{dx} = \frac{dP^{(m+1)}}{dp} = Q^{(m)} \text{ (n.° 8), } \quad \frac{ddZ}{2dx^2} = \frac{dQ^{(m)}}{2dp} = R^{(m-1)},$$

$$\frac{d^3Z}{2 \cdot 3dx^3} = \frac{dR^{(m-1)}}{3dp} = S^{(m-2)}$$

Dunque queste quantità $P^{(m+1)}$, $Q^{(m)}$, $R^{(m-1)}$, $S^{(m-2)}$, ec. al sono che i coefficienti della Equazione in y , ed essa sarà la seguente

$$(F) \quad P^{(m+1)} + Q^{(m)}y + R^{(m-1)}y^2 + S^{(m-2)}y^3 + \text{ec.} + Ay^m =$$

“Ciò essendo, per isciogliere il nostro Problema, altro dovremo, che determinare speditamente le quantità $P^{(m+1)}$, $R^{(m-1)}$, $S^{(m-2)}$, ec.

“A tal fine disposti in (IV) i coefficienti della (A) nello stesso del (I) (n.° 9), e determinate nella maniera tutte le quantità A , P' , P'' , ec. $P^{(m+1)}$, colloco in una terza linea sotto della P'' il coefficiente A , moltiplico questo per p , il prodotto $Ap = Q'p$ (2.° n.° 8) col sovrapposto P'' , e il risultato $Q'p + P''$, essendo pel cit.° (2.° n.° 8) $= Q''$, si ponga terza linea sotto del termine P'' . Si moltiplichi Q'' per p , si unisca con P''' , e pongasi sotto del P^{IV} il risultato $Q''p + P''' = Q'''$ (2.° n.° 8). Proseguendo così ad altre otterremo sotto del $P^{(m+1)}$ il risultato $Q^{(m)} \dots$

$$\begin{array}{rcl}
 A, B, C, D, E, F, & \text{ec. } V, \\
 A, P'', P''', P^{IV}, P^V, & \text{ec. } P^{(m)}, P^{(m+1)}, \\
 A, Q'', Q''', Q^{IV}, & \text{ec. } Q^{(m-1)}, Q^{(m)}, \\
 (IV) \quad A, R', R'', R''', & \text{ec. } R^{(m-2)}, R^{(m-1)}, \\
 & A, S'', & \text{ec. } S^{(m-3)}, S^{(m-2)}, \\
 & A, & \text{ec. } T^{(m-4)}, T^{(m-3)}, \\
 & & \text{ec.}
 \end{array}$$

"Nell' ultima $(m + 1)$ esima colonna verticale contenendosi i valori dei coefficienti della (F) , li sostituisco nella (F) medesima, e per tal guisa otterremo la trasformata richiesta con operazioni semplici, e facili ad eseguirsi, come ognuno può agevolmente vedere da se, e come apparisce dall' esempio seguente.

"Data la $4x^5 - 6x^4 + 3x^3 - 5x^2 - 4x + 8 = 0$, vogliasi trasformare in un' altra, nella quale sia $y = x - 6$. Opero nel modo ora esposto,

$$\begin{array}{rcccccc}
 4, & -6, & 3, & -5, & -4, & 8, & \\
 & 4, & 18, & 111, & 661, & 3962, & 23780 \\
 & & 4, & 42, & 363, & 2839, & 20996 \\
 & & & 4, & 66, & 759, & 7393 \\
 & & & & 4, & 90, & 1299 \\
 & & & & & 4, & 114 \\
 & & & & & & 4
 \end{array}$$

e otterremo la trasformata

$$4y^5 + 114y^4 + 1299y^3 + 7393y^2 + 20996y + 23780 = 0."$$

It appears from this that the computation demanded by Ruffini is identical with that in the familiar process of Horner. But the arrangement of the figures is different. The coefficients of the transformed equation appear, with Ruffini in the column on the extreme right, with Horner along a diagonal line slanting upward from left to right. It is not evident that, of the two arrangements, one possesses any marked superiority over the other. On pages 77-79 Ruffini computes the real root of Newton's cubic, $x^3 - 2x - 5 = 0$, to eight decimal places.

Ruffini in 1804 elaborated "Horner's method" of approximation to the roots of numerical equations with a clearness and thoroughness not surpassed in Horner's own exposition of 1819. In view of this fact, does not historical justice demand that the name of Ruffini be associated with that of Horner in

the designation of the Horner method"?

It should be stated of 1807,* explained into another whose resemblance to that of are to be diminished. But when a root is to ence: Budan does this by unity in each. I have written to Delambre, the following:†

"Troppo giuste sono le obiezioni di Budan di sciogliere la questione che avesse Ella avuta in mente sullo stesso argomento. Io ho già detto che da essa riportò gran parte della sua opera nel 1804 fuori della maniera di fare le tra decimali."

THE NEW HAV

The New Haven Colloquy
ERNEST JULIUS W
iversity Press, 1910

THE fifth Colloquy was held at New Haven under the auspices of Yale University in which recent progress in mathematics by men who have made the volume which contains the interest, particularly Professor Moore gives a form of G. H. Hardy termed "a form of G. H. Hardy outlines the point of

*F. D. Budan, *Nouv. Méthode*, Paris, 1807, p. 1.

† *Memorie della Società di Scienze*, XIV, 1906, p. 296.

projective differential geometry. Professor Mason treats a variety of boundary value problems.

I. *Introduction to a Form of General Analysis.*

It is obvious to those who have been following recent mathematical progress that, since the researches of Hill, Volterra, and Fredholm in the direction of extended linear systems of equations, mathematics has been in the way of a great development. That attitude of mind which conceives of the function as a generalized point, of the method of successive approximation as a Taylor's expansion in a function variable, of the calculus of variations as a limiting form of the ordinary algebraic problem of maxima and minima is now crystallizing into a new branch of mathematics under the leadership of Pincherle, Hadamard, Hilbert, Moore, and others. For this field Professor Moore proposes the term "General Analysis," defined (page 9) as "the theory of systems of classes of functions, functional operations, etc., involving at least one general variable on a general range." He has fixed attention on the most abstract aspect of this field by considering functions of an absolutely general variable. The nearest approach to a similar investigation is due to Fréchet (Paris thesis, 1906), who restricts himself to variables for which the notion of a limiting value is valid.

In the General Analysis we consider a class $\mathfrak{M}$ of real single valued functions ϕ_p of the variable p ; important illustrative cases are:

- I. $p = 1, 2, \dots, n$; $\phi_p \equiv (\phi_1, \phi_2, \dots, \phi_n)$, where $\phi_1, \dots, \phi_n$ are arbitrary real quantities.
- II. $p = 1, 2, \dots, n, \dots$; $\phi_p \equiv (\phi_1, \phi_2, \dots, \phi_n, \dots)$, where $\phi_1, \phi_2, \dots$ are restricted in that $\lim_{p=\infty} \phi_p = 0$.
- III. The same as II, except that the convergence restriction is now that $\sum_{p=1}^{\infty} \phi_p^2$ converges.
- IV. $0 \leq p \leq 1$; ϕ_p is any real continuous function of p .

The class $\mathfrak{M}$ is linear (L) if the sum of every two functions of $\mathfrak{M}$ or the product of one such function by any real constant is in $\mathfrak{M}$.

The class $\mathfrak{M}$ is closed (C) if the limit ϕ_p of a convergent sequence $\alpha_p, \beta_p, \dots$ of functions of $\mathfrak{M}$ is itself a function of $\mathfrak{M}$ whenever there exists a function λ_p of $\mathfrak{M}$ such that the difference between the successive members of the sequence and

the limit ϕ_p becomes and absolute value than $\epsilon\lambda_p$, which is a quantity. This mode of convergence as to the form convergence if the sequence

The class $\mathfrak{M}$ possesses the finite or infinite sequence α_p dominating function λ_p of $\mathfrak{M}$ such that the sequence does not exceed a constant multiple of λ_p for any p .

Finally if the absolute value of α_p is a function of $\mathfrak{M}$, that class

It is clear that in the case of sequences has the properties L , D will hold (cf. Theorem, I).

The property C in case that the limit of a uniform continuous functions is itself. II, C may be seen to hold. A sequence of sequences each successive sequences converging to a sequence, and furthermore the term difference from the limit is less than ϵ times the corresponding term of the limit sequence. The limit sequence converging to zero since its terms are less than ϵ times the sum of the absolute values of the two sequences converging to zero. and to demonstrate this from the above argument by replacing the condition 'with sum of squares' by the condition 'with sum of squares' note that the sum of squares is less than twice the sum of the squares of the sequences in $\mathfrak{M}$ by virtue of

$$(a + b)^2 \leq 2(a^2 + b^2)$$

The property D holds in case as the dominating function $M_1, M_2, \dots$ denote positive

*The word 'dominating' is used in the above sense.

the sequence of $\alpha_p, \beta_p, \dots$ respectively in absolute value, we may take

$$\lambda_p = \frac{|\alpha_p|}{M_1} + \frac{|\beta_p|}{2M_2} + \frac{|\gamma_p|}{4M_3} + \dots$$

since then clearly $\lim_{p \rightarrow \infty} \lambda_p = 0$, so that λ_p is in $\mathfrak{M}$ and also, by definition, will dominate the members of the sequence; in case III we may take λ_p so that

$$\lambda_p^2 = \frac{\alpha_p^2}{M_1} + \frac{\beta_p^2}{2M_2} + \frac{\gamma_p^2}{4M_3} + \dots,$$

where $M_1, M_2, \dots$ exceed the sum of the squares for the sequences $\alpha_p, \beta_p, \dots$ respectively, since then clearly $\sum_{p=1}^{\infty} \lambda_p^2$ is a convergent series and also λ_p is a dominating function for each sequence.

Suppose now that any class $\mathfrak{M}$ of functions be given which possesses the property D . We may extend the class $\mathfrak{M}$ by adding to it all finite sums of constant multiples of functions of $\mathfrak{M}$. Since by D we may find a dominating function μ_p of $\mathfrak{M}$ for the functions in this sum, every function of this extended linear class $\mathfrak{M}_L$ does not exceed in absolute value a constant multiple of some function of $\mathfrak{M}$, for example $m\mu_p$, where μ_p is chosen as indicated and m is a sufficiently large constant. Let us now add to $\mathfrak{M}_L$ the limit functions ϕ_p of every convergent sequence of functions of $\mathfrak{M}_L$ which is relatively uniformly convergent as to a function λ_p of $\mathfrak{M}_L$; such a sequence may be written in the form of an infinite series of functions of $\mathfrak{M}_L$

$$\phi_p = \alpha_p + \beta_p + \dots$$

since $\mathfrak{M}_L$ is linear. Inasmuch as λ_p does not exceed in absolute value some function of $\mathfrak{M}$, we may take the scale function λ_p to be in $\mathfrak{M}$. This extension of the class $\mathfrak{M}_L$ is called the $\star$ -extension of $\mathfrak{M}$ and is denoted by $\mathfrak{M}_\star$. Since the remainder after n terms in the above series may be taken not to exceed $|\lambda_p|$ itself, by taking n sufficiently large, and since we can find a dominating function for each of these n terms and for λ_p which belongs to $\mathfrak{M}$, the same argument that was made above shows that ϕ_p does not exceed in absolute value some constant multiple of a function of $\mathfrak{M}$. Hence $\mathfrak{M}_\star$ possesses the dominance property also, and the dominating function of any

sequence may be taken
apparent that $\mathfrak{M}_*$ is lin

A second extension (§
III, page 52). For
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By adding to $\mathfrak{M}$ its a
and then by extending
functions obtained from
character form a class $\mathfrak{L}$
*-extension of $\mathfrak{M}_{AL}$ is α
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 $\mathfrak{M}$ and have the same c

$$(\mathfrak{M}_{\#})_{\#} = [(\mathfrak{M}_{\#})_{AL}]_{*} =$$

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as to P is the greatest common subclass of the classes containing $\mathfrak{M}$ and having the property P .

Part I concludes with a proof of the complete independence of the properties A, L, C, D : that is, it is proved that all the $2^4 = 16$ conceivable combinations of these properties and their negatives are consistent.

In Part II are treated the composition of classes $\mathfrak{M}$ and $\mathfrak{M}'$ on the respective ranges p and p' . If both $\mathfrak{M}$ and $\mathfrak{M}'$ possess the property D , the product class $\mathfrak{M}\mathfrak{M}'$ on the product range pp obtained by multiplying any function ϕ_p of $\mathfrak{M}$ by any function $\phi_{p'}$ of $\mathfrak{M}'$, has also the property D : for from any sequence $\alpha_p, \alpha'_{p'}, \beta_p, \beta'_{p'}, \dots$ we deduce dominating functions λ_p for the sequence $\alpha_p, \beta_p, \dots$ and $\lambda'_{p'}$ for $\alpha'_{p'}, \beta'_{p'}, \dots$, and thus obtain a dominating function $\lambda_p \lambda'_{p'}$ in $\mathfrak{M}\mathfrak{M}'$ for the given sequence.

We may now obtain $(\mathfrak{M}\mathfrak{M}')_*$ which is readily shown to be the same as $(\mathfrak{M} * \mathfrak{M}')_*$, $(\mathfrak{M}\mathfrak{M}*)_*$ or $(\mathfrak{M} * \mathfrak{M}*)_*$ (cf. Theorem I, page 95). In fact the classes $\mathfrak{M} * \mathfrak{M}'$ and $\mathfrak{M}\mathfrak{M}*_*$ form part of $(\mathfrak{M}\mathfrak{M}')_*$ and contain $\mathfrak{M}\mathfrak{M}'$. Hence the statement must be true for the first two of the three classes since $(\mathfrak{M}\mathfrak{M}')_*$ is the same as its $*$ -extension. By a second extension as to p' one concludes that $\mathfrak{M} * \mathfrak{M}'$ belongs to $(\mathfrak{M}\mathfrak{M}')_*$ and at the same time contains of course $\mathfrak{M}\mathfrak{M}'$. Hence the statement is true for the third class. Similar results may now be given at once for the composition of more than two classes (cf. Theorems II, III, IV, page 95, 96).

In the special case where $\mathfrak{M}$ and $\mathfrak{M}'$ are the classes of all continuous functions of p and p' respectively, the class $(\mathfrak{M}\mathfrak{M}')_*$ is the class of all continuous functions of the two variables. This well known fact serves to orient one toward the remainder of Part II. The continuity in p, p' of a function $\phi_{pp'}$ of this class $(\mathfrak{M}\mathfrak{M}')_*$ is equivalent to the following two conditions (1) that $\phi_{pp'}$ is uniformly continuous in p , *uniformly over the range* p' ; (2) $\phi_{pp'}$ is continuous in p' for every p . Professor Moore treats properties B of $\mathfrak{M}$ such that for every $\mathfrak{M}'$ with properties L, C, D every function of the class $(\mathfrak{M}\mathfrak{M}')_*$ is in $\mathfrak{M}$ for every p' and has the property B uniformly, and is in $\mathfrak{M}'$ for every p (cf. §§ 57-65). Of course the precise kind of uniformity needs to be stated in each case. The fact that $(\mathfrak{M}\mathfrak{M}')_*$ in the particular case is the class of all continuous functions of pp suggests a consideration of the class $\mathfrak{F}$ of *all* functions of pp belonging to the class $\mathfrak{M}$ and having a property B uniformly for every p' and belonging to the class $\mathfrak{M}'$ for every p (cf. Theorem, page 109, Theorem I, page 110).

s taken to have a certain additional property Δ closely allied to the limit property of Fréchet, but of a *non-metric* character.* In the application of the General Analysis it is found necessary to have such properties, in order to secure a satisfactory characterization of functions of $(\mathcal{MM})_*$ on the range pp' .†

Professor Moore has found it possible to define the analog of the class of all continuous functions over ranges of this general description including the types presented in I–IV. The property of these ranges which is generalized is that of the denumerability in I, II, III and that of sequential halving of the interval in IV.

The generalization is obtained as follows: A development Δ of the range p is a sequence $\Delta^m (m = 1, 2, \dots)$ of systems of subclasses $\Delta^{mi} (i = 1, 2, \dots, l_m)$ of elements of the range, for a particular m giving the m th stage Δ^m of the development Δ . A function ϕ_p dominated by a function of $\mathcal{M}$ is said to have the property K_{12} relative to $\mathcal{M}$ and Δ (generalization of convergence and continuity) in case the difference between the values of ϕ_p at a fixed point p^{mi} and a variable point p in any one class Δ^{mi} of the m th stage tends to zero, as m increases, relatively uniformly to some function of $\mathcal{M}$, and at the same time ϕ_p itself tends to zero in the same manner for points p in no class Δ^{mi} of the m th stage. It is then easy to demonstrate that all functions of $\mathcal{M}_*$ possess the property K_{12} if those of $\mathcal{M}$ do, and also that in case $\mathcal{M}'$ is a second class of functions having the property K_{12} with respect to $\mathcal{M}'$ and Δ' , the set $\mathcal{MM}'$ possesses the property K_{12} with respect to the composite development $\Delta\Delta'$ (cf. Theorem II, page 135).

Suppose now that we have a set of functions of $\mathcal{M}$, say δ_p^{mi} , one for each subset of the development Δ of the range, such that (1) the sum of the absolute value of the functions $\delta_p^{mi} \phi_{p^{mi}}$ p^{mi} a representative point of Δ^{mi} and ϕ_p with the property K_{12} whose corresponding Δ^{mi} does not contain p becomes small relatively uniformly with respect to some function ϕ_{op} of $\mathcal{M}$ as i increases, (2) the sum of those δ_p^{mi} whose corresponding Δ^{mi} does contain p , as well as the sum of their absolute values, tends toward 1 uniformly as m increases. This set of functions

* Dr. T. H. Hildebrandt deals with the matter in his Chicago thesis of 910.

† See the article by Professor Moore in the *Atti* of the Rome International Congress, 1908, vol. 2, pp. 98–114.

have clearly the property that any such function ϕ_p may be written as

$$\lim_{m \rightarrow \infty} \sum_{\mathfrak{M}} \phi_{p^{m_i}} \delta_p^{m_i}.$$

In fact we have

$$|\phi_p - \sum_{\mathfrak{M}} \phi_{p^{m_i}} \delta_p^{m_i}| \leq |\phi_p - \sum_1 \phi_{p^{m_i}} \delta_p^{m_i}| + |\sum_2 \phi_{p^{m_i}} \delta_p^{m_i}|,$$

where the first and second summation are extended respectively to the functions $\delta_p^{m_i}$ whose Δ^{m_i} does and does not contain p . But by (1) the second term on the right is not greater than $\epsilon |\phi_{0p}|$, where ϵ is arbitrarily small for m large, and ϕ_{0p} is a function of $\mathfrak{M}$. Also the first term is not greater than

$$|\phi_p - \sum_1 \phi_{p^{m_i}} \delta_p^{m_i}| + |\sum_1 (\phi_p - \phi_{p^{m_i}}) \delta_p^{m_i}|.$$

By (2) the first member of this sum does not exceed $\epsilon |\phi_p|$ for m large if there are any terms in $\sum_1$, and by K_{12} does not exceed $\epsilon |\phi_{1p}|$ in the contrary case (p not in any Δ^{m_i}). The second term does not exceed $\epsilon |\phi_{2p}|$ by K_{12} and (2) or else is zero. Hence any function ϕ_p with the property K_{12} can be represented as stated and the convergence is relatively uniform with respect to some function of $\mathfrak{M}$ (namely with respect to any function which dominates ϕ_p , ϕ_{0p} , ϕ_{1p} , ϕ_{2p}), since $\mathfrak{M}$ possesses the property D . It follows that ϕ_p belongs to $\mathfrak{M}_*$ (cf. Theorem §I, page 140).

The simplest illustration of the above is afforded by II, III when

$$\delta_p^{m_i} = \begin{cases} 0, & p \neq l \\ 1, & p = l \end{cases} \quad \text{and} \quad l = 1, 2, \dots, m.$$

In case IV a developmental system is well known also.

The composition theory of developments Δ and of developmental systems $\delta_p^{m_i}$ is now finally considered; the functions of the composite developmental systems for any stage m are given by the products of the developmental functions of each of the classes for the same stage.

The notational scheme of the General Analysis makes possible a very abbreviated statement of the results. For example the last theorem dealt with above is written

$$\mathfrak{M}^{D_{K_{12}\Delta}} \supset \mathfrak{M}_* = [\text{all } \phi^{K_{12}\Delta}].$$

This affirms that if $\mathfrak{M}$ possesses the properties: D , the domin-

ance property ; K_{12} ,
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 Moore to the review

P. 128. A definitio
 lines 5 and 9 res]

$$(6) \frac{K'_{x'p}}{p' \cdot E}$$

$$(6) (K',$$

P. 129, ll. 3-5. E
 P. 143, ll. 16-26.

$$\theta_{\mathfrak{M}p} - \phi_p = \sum_k^* \phi_k$$

by suitable use of

$$(21) p^m \cdot (m \geq m_0$$

P. 147, l. 15. *After in fact, insert*

in case $\mathfrak{M}^{D_1 K_2}$ and $\mathfrak{M}^{D_1 K_2'}$,

P. 147, ll. 20–21. Insert the respective conditions

$D_1 K_2', D_1 K_2''; K_2', K_2'',$

on the classes $\mathfrak{M}', \mathfrak{M}''$ in the hypotheses of the propositions of lines 20; 21.

P. 148, l. 12. *For* $D\Delta$ *read* $DK_2^{\Delta}\Delta$.

P. 148, l. 25. *For* $LCDA$ *read* $LCDK_2^{\Delta}\Delta$.

II. *Projective Differential Geometry.*

In these lectures Professor Wilczynski presented an outline of some of the most important results of the projective differential geometry of curves and ruled surfaces, and at the same time gave an indication of the method of proof. A complete presentation of these results will be found in Professor Wilczynski's treatise on Projective Differential Geometry. For this reason the reviewer confines himself to a very brief synopsis of these readable lectures.

Projective differential geometry deals with the differential properties of geometrical configurations that are invariant under the projective group.

The curve in $(n-1)$ space presents itself as given by n homogeneous coordinates $y_1, \dots, y_n$, each a function of the parameter x . Any other set of such coordinates in the same parameter will be given by any n linearly independent combinations of these, aside from a multiplicative transformation. Hence the homogeneous linear differential equation of the n th order, a fundamental set of whose solutions is $y_1, \dots, y_n$ is the same as that given by any other projectively equivalent curve, or else is obtained from it by a multiplicative transformation of the dependent variable. If the factor is chosen so as to make the $(n-1)$ th derivative disappear, the ratios of the remaining coefficients will then be projective invariants of a given curve with given parameter and are called the semi-invariants. The semi-covariants are similarly defined.

The absolute invariants and covariants are further invariant and covariant under a transformation of the parameter and are readily obtainable from the semi-invariants and covariants.

The geometrical interpretation of these functions leads to the projective differential geometry of curves, and is not given in the lectures by Professor Wilczynski. His contribution in this direction has been to make systematic use of the elegant analytic instrument afforded by the differential equation, and thus to obtain Halphen's results for the plane and space, and in addition like results in a general space.

The projective theory of the ruled surfaces in space whose generators are the lines joining corresponding points of two curves C_y and C_z with coordinates

$$y_1, y_2, y_3, y_4 \text{ and } z_1, z_2, z_3, z_4$$

is then treated. The most general set of coordinates (aside from a factor) is obtained by taking the same four linearly independent linear combinations of the y 's and z 's. Thus we are led to consider

$$\sum c_y y_i \text{ and } \sum c_z z_i.$$

These functions form the general solution of a pair of ordinary linear differential equations

$$y'' + p_{11}y' + p_{12}z' + q_{11}y + q_{12}z = 0,$$

$$z'' + p_{21}y' + p_{22}z' + q_{21}y + q_{22}z = 0.$$

Conversely any set of four linearly independent particular solutions y, z of such equations yield always projectively equivalent ruled surfaces. These equations are fundamental for the theory which has been created by Professor Wilczynski.

A change of the dependent variables

$$\bar{y} = \alpha(x)y + \beta(x)z, \quad \bar{z} = \gamma(x)y + \delta(x)z$$

changes C_y and C_z into any other pair of curves $\bar{C}_y$ and $\bar{C}_z$ on the same ruled surface, and a change of independent variable $\bar{x} = f(x)$ changes the parameter on the curves in any given way. The invariants and covariants under these transformations are now obtained analytically and their interpretation leads to the geometrical results.

Since the space dual of a ruled surface is a second ruled surface one may expect a second pair of adjoint equations to be associated with the given one. The invariants are the same for

the adjoint equations as for the given equations, except in sign. The condition that these two sets of equations are the same leads to the theorem that a ruled surface is projectively equivalent to its dual only in case it is a quadric.

The asymptotic lines on ruled surfaces are clearly projectively invariant. The condition that the curves C_1 and C_2 are asymptotic lines is simply $p_{12} = p_{21} = 0$ and in this case the integral curves of the given equations form the asymptotic lines. A simple consequence is the theorem of Paul Serret: the cross ratio formed by the intersection of a moving generator with four fixed asymptotic lines is constant.

The totality of the tangents to the asymptotic lines along a given generator g form one set of rulings of an osculating hyperboloid of the ruled surface. The totality of generators of the hyperboloids thus constructed form a congruence Γ . One of the covariants P gives a unique second generator g' of the hyperboloid H of the same set as g and thus a second ruled surface varying arbitrarily with the choice of independent variable of which it serves as the image. This ruled surface is the derivative ruled surface S' of S with respect to x .

The flecnodal curves are two curves along which the tangents to the asymptotic lines have four points in common with the ruled surface. The flecnodal points are determined by the equation to zero of a certain quadratic covariant C . If the flecnodal points coincide, one of the invariants θ_1 vanishes. The developables of which the flecnodal curves are the cuspidal edges form the focal surfaces of the congruence Γ . The condition that another invariant Δ vanishes is that the ruled surface belongs to a linear complex.

In the final lecture there are given first some theorems concerning the derivative surface of S . The lecturer then passes on to develop the notion of the two complex points (arising from another quadratic covariant) on each generator which with the flecnodal points form a harmonic group. Lastly the theorems which state the extent to which the flecnodal curves may be arbitrarily assigned are given.

III. *Theory of Boundary Value Problems.*

Professor Mason considers a functional equation $f = g + Sf$ for x on a range R , where S is a linear operator such that (1) $S\phi$ is continuous when ϕ is continuous, (2) the series

$$\phi + S\phi + S^2\phi + \dots$$

converges uniformly in H
 S on the series is the same
 In this case S is suggestive
 the infinite series above for
 unique continuous solution

As a first application,
 linear differential equation
 The method does not depend
 approximation, save in form

The second application
 tion in two independent
 standard reduction of line
 second order to normal form
 bolic case, this form is

$$\frac{\delta^2 u}{\delta x \delta y} = a$$

Professor Mason treats
 problem: Given a rectangle
 upon it two curves $C_x: y$
 $(y_1 \leq y \leq y_2)$ where ϕ, ψ, z
 to determine a solution u
 $\delta u / \delta y$ take assigned values
 respectively. Professor Mason
 exists. The critical point
 factorially presented than
 operator S which appears
 "majorant operator."

This boundary value problem
 considered for this type of
 in which the curves reduce
 and that in which u and
 one and the same monotonic

The simplest equation
 equation in the plane, is
 Neumann is employed to
 a convex region bounded
 which takes given values
 the operator S is

$$S(\phi) = \frac{1}{\pi} \int_0^t \left\{ [\phi(s) \right.$$

which is also proved to be convergent. A short review of the more general methods which permit one to consider regions not convex is given.

The Green's function $G(x, y : \xi, \eta)$ for the potential equation is exhibited in the customary way by means of the result just stated. Then follows the interesting question of the existence of doubly periodic solutions u of $\Delta u = f(x, y)$, where $f(x, y)$ is doubly periodic in x, y with periods a, b . The real part of $\log \sigma(z - \xi) - \log \sigma(z - \eta)$, where σ is the Weierstrassian σ -function and $z = x + iy$, furnishes a solution of $\Delta u = 0$ analytic at all points of the period rectangle save for an infinity at $z = \xi$ and $z = \eta$ like $\pm \log r$ (r the distance from x, y to ξ, η) and furthermore is doubly periodic except for simple linear terms. By a slight modification destroying these terms the Green's function for the equation $\Delta u = 0$ and the boundary conditions $u(\alpha) = u(\alpha + a)$, $u(\beta) = u(\beta + b)$ are obtained. The necessary and sufficient condition for a doubly periodic solution is found by Professor Mason to be that the integral of f over the rectangle vanishes.* The proof depends on the properties of the Green's function.

It is next shown that a solution of

$$\Delta u + cu = f$$

taking assigned values on the boundary of a region R exists if R is taken sufficiently small. The operator S is obtained by means of the Green's function of $\Delta u = 0$.

Attention is then called by the lecturer to the recent work of Serge Bernstein, in which are proved simple criteria for the analytic or non-analytic nature of solutions of partial differential equations of the second order.

In conclusion, the lecturer considers the boundary value problem attaching to the equation and condition

$$y'' + \lambda A(x)y = 0, \quad y(a) = y(b) = 0, \quad (\lambda \text{ a parameter}),$$

as it arises from the consideration of the transverse vibrations of a stretched string. The method of the existence proof depends on the solution of a certain minimum problem, and in order to

* See an article in the *Transactions of the American Mathematical Society*, vol. 6 (1905), p. 159.

carry through this proof the g

$$y'' + \lambda A(x)y =$$

is first derived. The minimum

is to minimize $\int_a^b y'^2 dx$ under the

$$\int_a^b y y dx = 0 \quad (i = 1, 2, \dots, n),$$

belonging to $\lambda_1, \dots, \lambda_n$. The

magnitude the positive values

ber if ϕ is anywhere positive

values of λ will be obtained if

A formal expansion of an ar

$$f = \sum_{-\infty}^{+\infty} c_n y_n$$

and Professor Mason states t

holds if f vanishes at a and b ,

tive continuous save at a finite

however contains an error.†

SHORTER

Vorlesungen über Algebra.

gegeben vom Mathematischen

Lehrstuhl. Leipzig und Berlin,

366 pp.

THAT Professor Bauer's lec

honor of his 80th birthday

Munich in 1903, are destined

years, seems to be evidenced

became necessary in 1910, se

and four years after Profess

on April 3, 1906.

* See an article in the *Transactions*

† At bottom of p. 219 it is neces
inequality

$$(\phi_n, m)^2 \leq 2 \sum_{i=1}^m$$

by a multiplier $m - n$. This error a

A fine spirit of veneration for the deceased teacher and friend breathes from the preface for this second edition, written by Professor Karl Doehlemann, who has now taken full charge of the book. It is clear that Professor Bauer, whose portrait appears opposite the title page, was much respected by those who worked under him and with him.

As the first edition of these lectures was quite fully reviewed by Professor Dickson on pages 257-260 of volume 10 of the *BULLETIN*, the task of the present reviewer may be limited to a discussion of the points of difference between the two editions. In Part I, devoted to the general properties of algebraic equations, the most important change consists in the substitution of Ulherr's proof of the fundamental theorem of algebra for the proof of Argand, which only showed that if $f(z) \neq 0$, a complex number h could be found, such that $|f(z+h)| < |f(z)|$. Ulherr's proof makes use of the fact, that when z describes a circle about the origin in the z -plane, $w = f(z)$ describes a curve in the w -plane which goes about the origin n times, if $f(z)$ is of degree n .

The notion of a domain of rationality, formerly taken up in the chapter on the Galois theory, is now introduced in Chapter IV, as are also the notions of reducibility and irreducibility of functions and of equations.

Part II, which treats the algebraic solution of equations, has undergone greater changes than the other parts. The chapter on the Galois theory of algebraic equations has been omitted and a new chapter has been inserted on Abel's theorems concerning the general representation of algebraic functions. This makes possible a more rigorous presentation than was formerly possible of the Abel-Wantzel theorem that equations of degree higher than the fourth are not in general solvable by radicals. In this new Chapter XV, which follows exactly the lines of the classical treatise of Serret, one feels a dissimilarity from the older chapters in the complete absence of illustrative examples, so abundant elsewhere. The book, as stated in the preface, does not take the ultra-modern point of view, and is intended for students who for the first time are making a systematic study of the science of algebra. In this chapter, where they are getting a first introduction to the more difficult parts of the subject, a few well chosen examples would help them to keep the general statements connected with the concrete situations of which they are the abstraction.

Part III, in which the
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to the solution of a system
in $n + 1$ variables. By i
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results are stated more cor
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reviewer that still further
made, as is shown for in
Introduction to Higher A

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It is remarkable, that w
first edition have been corr
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Théorie mathématique des
E. PETIT. Paris, Doin

THE series of books c
Encyclopédie Scientifique
this country, yet the enterp

triously carried on under the editorship of Dr. Toulouse of the *École des Hautes-Études*. The handy little volumes are "in 18 *jésus cartonnés*," which is better understood in the original than in a translation, contain about 400 pages, and sell uniformly at 5 francs. The individual volumes would doubtless carry considerable weight by themselves; but their value and influence is much enhanced by the fact that each fits into its appropriate nook in the vast *Encyclopédie Scientifique*, which is divided into 40 sections and which will contain about 1000 volumes in all. The fact that the volumes are printed separately makes it possible to keep them revised and up-to-date; the fact that they belong to a series serves as a check on unnecessary repetition; the limited but not too limited size of the volumes allows each to give a good idea of its field without developing into a ponderous treatise. The sub-editors under Toulouse are Painlevé for the philosophy of science, Drach for mathematics and for mechanics, D'Ocagne for applied mathematics and for applied mechanics; and the supervisors in other fields are equally qualified for their tasks.

Richard and Petit, who write the volume on insurance, are actuaries and graduates of the *École Polytechnique*. They have produced a book in the spirit of the whole series, readable, yet not merely popular. The theory of insurance is developed in general and is applied to the discussion of numerous kinds of policies. The standard mortality tables are those compulsory in France; the types of policy treated in greatest detail are those most in vogue in France. This is but natural, and naturally the book will be most serviceable in France. But the foundation of the work is so carefully laid that it may be of value in any country. The authors do not overlook the fact that in several instances the insurance companies find it necessary for business reasons to adopt a procedure not wholly in accord with that which the theory would indicate as most rational. Unfortunately insurance has to contend with the psychology of the insured and insurable. This is nowise more evident than from the fact that the French government, which for forty years has maintained an insurance department guaranteed by the state, with smaller premiums than the private companies, and with no requirement for a medical examination, has written only about a million and a half dollars of insurance in the whole period! This cannot be attributed to partiality against the state; for the government old-age pensions or insurance are

a great success, the receipts for million dollars. The attitude of of practical as well as scientific

NO

PROFESSOR KLEIN'S Evans Mathematics, which have for have been republished by the SOCIETY and are now on sale at five cents per copy, postage free from the original plates, with A brief preface by Professor W The volumes are bound like t addressed to the American M 116th Street, New York, N. Y.

THE April number (volume *tions of the American Mathemat* ing papers: "Biorthogonal syst PELL; "Applications of biorth the theory of integral equations, uniform convergence of the dev by C. N. MOORE; "Determin lar linear groups," by H. H. M linear difference equations," by

THE April number (volume : *Journal of Mathematics* contains four or more homogeneous lines of the second order," by C. H. lines in space of four dimension geometry of the circle in space MOORE; "On the geometry of reference to osculating circles, "Binary modular groups and the SON; "The group of turns an turbines," by EDWARD KASNE

AFTER the completion of the *Mathematics* will be published u University instead of Harvard l

editorial staff will consist of Professors ORMOND STONE, of the University of Virginia, MAXIME BÔCHER, of Harvard University, and G. D. BIRKHOFF, L. P. EISENHART, E. SWIFT, OSWALD VEBLEN, and J. H. MACLAGAN-WEDDERBURN, of Princeton University. Correspondence concerning the new volume should be addressed to the *Annals of Mathematics*, Princeton University, Princeton, N. J.

At the meeting of the London mathematical society held on March 9 the following papers were read: By G. B. MATHEWS, "On the reduction and classification of binary cubic forms which have a negative determinant"; by A. E. H. LOVE, "The theory of the transmission of earthquake waves"; by P. A. MACMAHON, "The theory of partitions."

THE next annual meeting of the Deutsche Mathematiker-Vereinigung will be held at Karlsruhe in affiliation with the eighty-third convention of the society of German naturalists and physicians during the week beginning September 24. Titles and abstracts of papers to be presented should be sent to the secretary, Professor A. KRAZER, Westendstrasse 57, Karlsruhe, before May 15.

THE arrangements for the next International congress of mathematicians, to be held at Cambridge, England in August, 1912, are well under way and it is expected that the details will be announced shortly.

THE eminent mathematician, Professor GASTON DARBOUX, of the University of Paris, is about to complete his fiftieth year of service as a teacher in the system of public instruction of France. For more than twenty-five years he has been a member of the Academy of Sciences and for the past ten years he has been its permanent Secretary. It is proposed by a large international group of his mathematical co-workers, friends, and former pupils to commemorate this anniversary by presenting to Professor Darboux a gold medal bearing his portrait, and an appropriate address signed by the participants. All mathematicians are invited to share in rendering this honor to Professor Darboux. Copies of the medal, in reduced size, will be struck. Subscribers of twenty-five francs will receive a copy in bronze, subscribers of fifty francs a copy in silver. Subscriptions should be sent to Professor Cl. GUICHARD, secretary of the Faculté des Sciences.

THE rector and faculties of the Università that a commemorative tablet to the late CAPELLI be placed in the Sala Battaglini. Participants in the movement are invited to applications to the director of the mathematical Ernesto Pascal, Viale Elena 24, Naples.

THE subject of the Adams prize for 1912 is "The theory of radiation."

THE royal academy of sciences of Belgium has proposed the following prize problem for 1912: "To synthesize the investigations made in the calculus of variations since 1850." Competing memoirs should be written in Flemish and sent to the permanent secretariat before August 1, 1912.

THE following courses in mathematics are given at Cornell University for the year 1911-1912:

CORNELL UNIVERSITY.—By Professor J. H. VAN VLIET: Mathematical physics, three hours.—By Professor J. H. VAN VLIET: Teachers' course, three hours.—By Professor J. H. VAN VLIET: Elliptic functions, three hours.—By Professor J. H. VAN VLIET: Projective geometry, three hours.—By Professor J. H. VAN VLIET: Mechanics, three hours.—By Professor J. H. VAN VLIET: Theory of numbers, three hours.—By Professor J. H. VAN VLIET: Theory of functions (second term), three hours.—By Professor J. H. VAN VLIET: Theory of functions of a real variable, three hours.—By Dr. C. F. CRAIG: Algebraic curves, three hours.—By Dr. F. W. OWENS: Differential equations, three hours.—By J. V. MCKELVEY: Analytic geometry, three hours.—By L. L. SILVERMAN: Infinite series, three hours.—By Dr. F. W. OWENS: Algebra, three hours (second term).—By Dr. F. W. OWENS: Differential equations of mathematical physics, three hours.—By Dr. E. J. MILES: Advanced calculus, three hours.

PRINCETON UNIVERSITY.—By Professor J. H. VAN VLIET: Theory of elimination, three hours (first term).—By Professor J. H. VAN VLIET: Infinitesimal geometry, three hours.—By Professor J. H. VAN VLIET: Projective geometry, three hours.—By Professor J. H. VAN VLIET: Mechanics, three hours; Partial differential equations, three hours (first term); Vector analysis, three hours.—By Professor O. VEBLEN: Projective geometry, three hours.

Theory of functions of real variables, three hours.—By Professor G. D. BIRKHOFF: Analysis, three hours; Linear differential equations, three hours.—By Professor W. GILLESPIE: Theory of substitutions, three hours (first term).—By Professor J. G. HUN: Analytic projective geometry, three hours (second term).—By Professor E. SWIFT: Differential equations, three hours; Calculus of variations, three hours (second term).—By Professor J. H. McL. WEDDERBURN: Theory of functions of a complex variable, three hours.

THE following courses in the German universities will be given during the present summer semester:

UNIVERSITY OF BERLIN.—By Professor H. A. SCHWARZ: Integral calculus, with exercises, six hours; Selected chapters of elliptic functions, two hours; Calculus of variations, four hours; Colloquium, two hours; Seminar, two hours.—By Professor G. FROBENIUS: Theory of algebraic equations, four hours; Seminar, two hours.—By Professor F. SCHOTTKY: Theory of functions, II, four hours; Theory of curvature of curves and surfaces, four hours; Seminar, two hours.—By Professor G. HETTNER: Introduction to the theory of determinants, two hours.—By Professor J. KNOBLAUCH: Analytic geometry, four hours; Theory of space curves, II, one hour; Theory of surfaces, II, four hours.—By Professor R. LEHMANN-FILHFS: Differential calculus, four hours.—By Professor I. SCHUR: Ordinary differential equations, four hours; Introduction to the theory of functions, four hours.

UNIVERSITY OF BONN.—By Professor E. STUDY: Introduction to differential geometry, four hours; Seminar, two hours.—By Professor F. LONDON: Calculus, with exercises, four hours; Axonometry and perspective, two hours; Seminar, two hours.—By Professor F. HAUSDORFF: Theory of differential equations, four hours.—By Dr. J. O. MÜLLER: Calculus of variations, three hours.—By Dr. W. BLASCHKE: Algebraic equations, four hours.

UNIVERSITY OF GÖTTINGEN.—By Professor F. KLEIN: Introduction to differential and integral calculus, four hours; Seminar, two hours.—By Professor D. HILBERT: Mechanics of the continuum, four hours; Seminar, two hours.—By Professor E. LANDAU: Elements of the theory of numbers, four hours; Seminar, two hours.—By Professor C. RUNGE: Num-

erical calculation, six hours; Photogrammetry, By Professor F. BERNSTEIN: Theory of probabilities, two hours; Calculus of insurance, with exercises, two hours. — By Dr. O. TOEPLITZ: Theory of functions of a complex variable, two hours. — By Dr. H. WEYL: Analytic geometry, two hours. — By Dr. A. HAAR: Differential equations, two hours.

UNIVERSITY OF LEIPZIG.—By Professor K. VON KÖNIG: Projective geometry, two hours; Plane analytic geometry, two hours; Conic sections and quadric surfaces from the point of view of invariants, two hours; Seminar, two hours.—By Professor O. HÖLDER: Algebraic equations, two hours; Theory of functions of a complex variable, four hours; Seminar, two hours.—By Professor G. HERGLOTZ: Selected problems in calculus, one hour; Boundary values in the theory of functions, two hours; Seminar, two hours.—By Professor E. SCHROEDER: Ordinary differential equations, with exercises, two hours; Algebraic analysis, two hours.

UNIVERSITY OF MUNICH.—By Professor F. SCHUBERT: Analytic geometry of space, five hours; Elliptic functions, two hours; Seminar, two hours.—By Professor A. MARKOFF: Introduction to the theory of partial differential equations, two hours; Introduction to the theory of invariants, two hours; Seminar, two hours.—By Professor A. PRINGSHEIM: Calculus, five hours; Geometric applications of calculus, two hours.—By Professor H. BRUNN: Higher mathematics, three hours.—By Professor E. MANN: Descriptive geometry, with exercises, six hours; Algebraic constructions, with exercises, five hours.—By Professor F. HARTOGS: Algebra, II, four hours; Algebraic geometry, five hours.—By Professor L. v. SEELINGER: Theories of probabilities and least squares, four hours.—By Professor E. MERZFELD: Mechanics of continua with applications, two hours.

UNIVERSITY OF STRASSBURG.—By Professor E. BIEBERSCHWILDER: Definite integrals and introduction to the theory of functions, four hours; Theory of numbers, two hours; Seminar, two hours.—By Professor F. SOEUR: General theory of curves, four hours; Introduction to vector analysis, two hours; Seminar, two hours.—By Professor J. WELLSTEIN: Integral calculus, three hours; Theory of invariants, three hours; Seminar, two hours.—By Professor R. E. v. MISES: Descriptive geometry, two hours.

etry, four hours; with exercises, four hours; Theory of probabilities, two hours; Seminar, two hours.—By Professor P. EPSTEIN: Foundations of analysis, two hours.—By Professor M. SIMON: History of mathematics during the renaissance, two hours.

THE number of students of mathematics in the Prussian universities has increased from 1,440 in 1907-08 to 2,040 in 1910-11.

PROFESSOR TH. VAHLEN, of the University of Greifswald, has been promoted to a full professorship of mathematics.

DR. P. J. HEADWOOD has been appointed professor of mathematics at Durham University.

MR. R. D. CARMICHAEL, of Princeton University, has been appointed assistant professor of mathematics at Indiana University.

PROFESSOR J. ROSANES, of the University of Breslau, has retired from active service.

PROFESSOR F. A. SHERMAN, of Dartmouth College, will retire from active service at the close of the present academic year.

MR. A. R. MAXSON, for six years instructor in mathematics in Columbia University, died April 13, in his thirtieth year. He had been a member of the American Mathematical Society since 1905.

Recent catalogues of second hand mathematical books: A. Hermann et Fils, Paris, 6 rue de la Sorbonne, mathematics and sciences, about 150 titles in mathematics.—K. F. Koehler, Leipzig, Kurprinzstrasse 6, catalogue 586, mathematics, about 1200 titles.—G. E. Stechert, New York, 151 West 25th Street, catalogue 28, exact sciences, about 400 titles in mathematics.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

- COFFIN (J. G.). Vector analysis. An introduction to vector-methods and their various applications to physics and mathematics. New York, Wiley, 1911. 12mo. 19+248 pp. Cloth. \$2.50
- FABRY (E.). Théorie des séries à termes constants. Applications aux calculs numériques. Paris, Hermann, 1911. 8vo. 203 pp. Fr. 6.50
- . Traité de mathématiques générales, à l'usage des chimistes, physiciens, ingénieurs. Avec une préface de G. Darboux. 2e édition, revue et augmentée. Paris, Hermann, 1911. 8vo. 10+462 pp. Fr. 9.00
- FISCHER (P. B.). Koordinatensysteme. (Sammlung Göschen, 507.) Leipzig, Göschen, 1911. 8vo. 125 pp. Cloth. M. 0.80
- FRÉCHET (M.). Les fonctions d'une infinité de variables. Paris, Imprimerie nationale, 1910. 8vo. 8 pp.
- GRASSMANN (H.). Gesammelte mathematische und physikalische Werke. Unter Mitwirkung von J. Lüroth, E. Study, J. Grassmann, H. Grassmann d. J., G. Scheffers herausgegeben von F. Engel. 3ter Band, 1ter Teil. Leipzig, Teubner, 1911. 8vo. 5+353 pp. M. 18.00
- HACK (F.). Wahrscheinlichkeitsrechnung. (Sammlung Göschen, 508.) Leipzig, Göschen, 1911. 8vo. 123 pp. Cloth. M. 0.80
- HAMEL (A.). Petit traité d'analyse accompagné d'exercices pratiques. 4e édition, entièrement refondue, conforme à l'arrêté du 25 juillet 1910. Paris, Hatier, 18mo. 104 pp.
- JUNKER (F.). Repetitorium und Aufgabensammlung zur Differentialrechnung. (Sammlung Göschen, 146.) 3te, verbesserte Auflage. Leipzig, Göschen, 1911. 8vo. 129 pp. Cloth. M. 0.80
- KLEIN (F.). The Evanston Colloquium lectures on mathematics, delivered at Northwestern University Aug. 28 to Sept. 9, 1893. Reported by Alexander Ziwet. 2nd edition. New York, American Mathematical Society, 1911. 8vo. 11 + 109 pp. Cloth. \$0.75
- LEVY (A.). See SOMMER (J.).
- LORIA (G.). Spezielle algebraische und transzendente ebene Kurven. Deutsch von F. Schütte. 2te Auflage. 2ter Band: Die transzendenten und die abgeleiteten Kurven. (Teubner's Sammlung, Band V, 2.) Leipzig, Teubner, 1911. 8vo. 8+384 pp. Cloth. M. 14.00
- MEISER (W.). Lösungen zu Aufgaben aus der algebraischen Analysis. Nürnberg, Korn, 1910. 8vo. 3+66 pp. M. 2.00
- METTLER (H.). Graphische Rechnungsmethoden im Dienste der Naturwissenschaften. Zürich, 1910. 71 pp. Cloth. M. 2.00
- MINKOWSKI (H.). Gesammelte Abhandlungen. Unter Mitwirkung von A. Speiser und H. Weyl herausgegeben von D. Hilbert. 1ter Band. Leipzig, Teubner, 1911.
- NAKAGAWA (S.). Miscellanea aus dem Gebiete der hyperbolischen Geometrie. Tokyo, Journ. Coll. Sc., 1910. 4to. 68 pp. M. 2.40
- PACI (E.). Sulla integrazione di un'equazione di Riemann. Palermo, Virzi, 1910. 8vo. 35 pp.

RABICH (O.). Die Kegelschnitte im Lehrplane der Seminarien. Eisenach, 1910. 8vo. 24 pp.

SCHÜTTE (F.). See LORIA (G.).

SOMMER (J.). Introduction à la théorie des nombres algébriques. Edition française revue et augmentée. Traduit de l'allemande par A. Levy, avec préface de J. Hadamard. Paris, Hermann, 1911. 8vo. 10+376 pp. Fr. 15.00

VINOT (J.). Récréations mathématiques. Questions curieuses et utiles extraites des auteurs anciens et modernes. 6e édition. Paris, Larousse, 1911. 8vo. 215 pp.

WIPPER (J.). 46 Beweise des pythagoraischen Lehrsatzes nebst kurzen biographischen Mitteilungen. 2te Auflage. Berlin, Barsdorf, 1911. 8vo. 51 pp. M. 1.50

II. ELEMENTARY MATHEMATICS.

ABHANDLUNGEN über die Reform des mathematischen Unterrichts in Ungarn. Nach dem ungarischen Original unter Mitwirkung von M. Balog und J. Rados deutsch herausgegeben von E. Beke und S. Mikola. Leipzig, Teubner, 1911. 8vo. 6+160 pp. M. 4.00

BARLOW's tables of squares, cubes, square roots, cube roots, reciprocals of all integer numbers up to 10,000. London, Spon. 8vo. 208 pp. 4s.

BASSANI (A.). See LAZZERI (G.).

BEHRENDSEN (O.) und GÖTTING (E.). Lehrbuch der Mathematik für höhere Mädchen-Bildungsanstalten. Nach modernen Grundsätzen bearbeitet. 1ter Teil. 2te Auflage. Leipzig, Teubner, 1911. 8vo. 8+348 pp. Cloth. M. 3.00

BIELER (A.). Lehr- und Übungsbuch der Raumlehre für Knaben-Mittelschulen. 2te, nach den Lehrplänen von 1910 umgearbeitete Auflage. Leipzig, Teubner, 1911. 8vo. 7+176 pp. Cloth. M. 2.00

BORGOGNO (G.). Compendio di aritmetica, geometria e di sistema metrico decimale. 4ta edizione, riveduta ed accresciuta. Torino, Paravia, 1910. 16mo. 122 pp. L. 0.80

COLAW (J. M.). Elementary algebra. Richmond, Va., Johnson, 1911. 12mo. 368 pp. Half leather. \$1.00

DOUGLAS's two-color logs, anti-logs and mathematical tables. Prepared to meet the requirements of the leaving certificate examinations. London, Simpkin, 1911. 8vo. Sewed. 4d.

FÄRBER (C.). Arithmetik. (Die Grundlehren der Arithmetik und Algebra, bearbeitet von E. Netto und C. Färber. 1ter Band.) Leipzig, Teubner, 1911. 8vo. 15+410 pp. Cloth. M. 9.00

FREUD (P.). Die mathematischen Schulbücher an den Mittelschulen und verwandten Anstalten. (Berichte über den mathematischen Unterricht in Oesterreich, 6tes Heft.) Wien, Hölder, 1910. 8vo. 3+53 pp. M. 1.20

FROUMENTY (M.). See PHILIPPE (P.).

GAJDECZKA (J.). Prüfungsfragen aus der Mathematik. 3te, neubearbeitete Auflage der "Maturitätsprüfungsfragen." Wien, Deuticke, 1910. 8vo. 3+84 pp. M. 1.40

- GAMBIOLI (D.). *Geometria elementare*. Roma, Bertero, 1910. 8vo. 8+461 pp. L. 3.25
- GODFREY (C.) and SIDDONS (A. W.). *Elementary geometry, practical and theoretical, together with solid geometry*. London, Cambridge University Press, 1911. 8vo. 134 pp. 4s. 6d.
- . *Solid Geometry*. London, Cambridge University Press, 1911. 8vo. 120 pp. Sewed. 1s. 6d.
- GÖTTING (E.). See BEHRENDSEN (O.).
- HERTING (G.). I: *Von Strecke, Quadrat und Würfel zum bestimmten Integral*. Leipzig, Teubner, 1910. 8vo. 4+135 pp. Cloth. M. 2.80
- HOČVAR (F.). *Lehr- und Übungsbuch der Geometrie für Gymnasien und Realgymnasien*. 7te Auflage. Wien, Tempsky, 1911. Kr. 2.60
- HOWE (G.). *Mathematics for the practical man, explaining simply and quickly all the elements of algebra, geometry, trigonometry, logarithms, coordinate geometry, calculus*. New York, Van Nostrand, 1911. 12mo. 5+143 pp. \$1.25
- HUPE (A.). See MÜLLER (H.).
- KIND (H. W.). *The model practical mensuration. Based upon elementary geometric constructions arranged to suit the present gradation of school work*. New York, Hinds, 1911. 12mo. 89 pp. \$0.50
- LAAGER (F.). *Vereinfachter Lehrgang der Elemente der Trigonometrie*. Zurich, Speidel, 1911. 8vo. 40 pp. M. 1.20
- LANGEL (W.). See NIELSEN (C.).
- LAZZERI (G.) und BASSANI (A.). *Elemente der Geometrie. Unter Verschmelzung von ebener und räumlicher Geometrie. Aus dem Italienischen von P. Treutlein übersetzt*. Leipzig, Teubner, 1911. 8vo. 16+491 pp. Cloth. M. 14.00
- LUCHERINI (R.). *Fascicoli di calcolo per le scuole elementari. Nuova edizione, corretta e modificata*. Firenze, Bemporad, 1910. 16mo. 136 pp. L. 0.65
- MATHEMATICAL questions and solutions. Volume 18. New series. London, Hodgson, 1911. 8vo. 6s. 6d.
- MATRICULATION mathematics papers. Being the papers in elementary mathematics, set at the matriculation examination of the University of London, from June, 1903 to January, 1911. (University tutorial series.) London, Clive, 1911. 8vo. Sewed. 1s. 6d.
- MATRICULATION model answers, mathematics. London University matriculation papers in mathematics from September, 1907 to January, 1911. (University tutorial series.) London, Clive, 1911. 8vo. Sewed. 2s.
- MILLIS (J. F.). See STONE (J. C.).
- MÖNKEMEYER (K.) und RÜSEWALD (K.). *Lehr- und Übungsbuch der Mathematik für höhere Mädchenschulen. Auf Grund der neuen Lehrpläne bearbeitet*. 1tes, 2tes und 3tes Heft. Leipzig, Quelle und Meyer, 1911. 8vo. 45, 66 and 63 pp., respectively. M. 2.40
- MORTZ (R. E.). *College mathematics notebook for classes in advanced algebra, trigonometry, analytic geometry and calculus*. Boston, Ginn, 1911. 4to. \$0.80

- MÜLLER (H.). Die Mathematik auf den Gymnasien und Realschulen. Ausgabe B, unter Mitwirkung von A. Hupe. 2ter Teil: Die Oberstufe. 1te Abteilung: Planimetrie, Algebra, Trigonometrie und Stereometrie. 4te Auflage. Leipzig, Teubner, 1910. 8vo. 12+308 pp. Cloth. M. 3.00
- MÜLLER (O.). Tavole di logaritmi con cinque decimali. 11a edizione, aumentata delle tavole dei logaritmi d'addizione e sottrazione, per cura di M. Rajna. (Manuali Hoepli, XLI.) Milano, Hoepli, 1911. 16mo. 36+191 pp. L. 1.50
- NEVEU (H.). Cours d'algèbre théorique et pratique, suivi de notions de trigonométrie. 5e édition, conforme aux programmes de 26 Juillet 1909. Paris, Masson, 1911. 16mo. 6+480 pp. Fr. 3.00
- NEVEU (H.) et BELLENGER (H.). Cours de géométrie théorique et pratique. 3e année. Paris, Masson, 1911. 16mo. 403 pp. Fr. 3.00
- NIELSEN (C.) und LÄNGEL (W.). Planimetrie und Stereometrie für Landwirtschaftsschulen. Berlin, Parey, 1911. 8vo. 7+159 pp. Cloth. M. 2.50
- PHILIPPE (P.) et FROUMENTY (M.). Cours de géométrie, à l'usage des écoles pratiques de commerce et d'industrie. Tome 1er: 1re année. Tome 2: 2e année. Paris, Dunod et Pinat, 1910. 8vo. 244+245 pp. Fr. 8.00
- PINCHERLE (S.). Esercizi sull'algebra elementare. 2a edizione. (Manuali Hoepli, serie scientifica, n° 206.) Milano, Hoepli, 1911. 24mo. 6+136 pp. L. 1.50
- ROBINSON (A. D.). The arithmetic help. For pupils, teachers, business men. Akron, O., Werner, 1911. 8vo. 515 pp. \$1.00
- RÜSEWALD (K.). See MÖNKEMEYER (K.).
- SCOCCIANTI (F.). Compendio di trigonometria piana. Viterbo, Agnossotti, 1910. 8vo. 27 pp. L. 0.30
- SPERA (S.). Elementi di algebra, per le scuole tecniche. Milano, Operai, 1911. 16mo. 94pp. L. 1.25
- STONE (J. C.) and MILLIS (J. F.). Plane and solid geometry. Boston, Sanborn, 1911. 8vo. 400 pp. \$1.30
- . Plane geometry. Boston, Sanborn, 1911. 8vo. 260 pp. \$0.85
- . Solid geometry. Boston, Sanborn, 1911. 8vo. 140 pp. \$0.85

III. APPLIED MATHEMATICS.

- ANDOYER (H.). Cours d'astronomie. 1re partie: Astronomie théorique. 2e édition, entièrement refondue. Paris, Hermann, 1911. 8vo. 6+384 pp. Fr. 12.00
- ANGELINI (S.). Sull'ufficio dei potenziali ritardati nelle teorie elettromagnetiche. Milano, Albrighi, 1911. 8vo. 6 pp.
- BISESTI (P.). Computisteria e ragioneria per le scuole tecniche e commerciali. Abbiategrosso, Bollini, 1911. 8vo. 429 pp. L. 3.80
- BLANCARNOUX (P.). Nouveau manuel complet de l'ajusteur-mécanicien, etc. Tome 1er: Formules élémentaires. Tome 2: Notes pratiques d'ajustage mécanique. Paris, Mulo, 1911. 18mo. Tome 1er, 7+245 pp. Tome 2, 260 pp. Fr. 6.00
- CAPART (G.). See VIEWEGER (H.).

- CORNUT, TEXTIER et DUTHEIL. Notions élémentaires de mécanique générale. 2e année: Statique, dynamique. Paris, Nathan, 1910. 8vo. 240 pp.
- DAVID (M.). Hilfbuch für die Berechnung von Gewölben mit parabolischer Achse, auf Grundlage der Elastizitätstheorie. Wien, Lehmann und Wentzel, 1910. 8vo. 50 pp. M. 3.00
- DENIS (P. J.). Moteurs électriques à courant continu et leurs dispositifs de commande. Propriétés dynamiques. Paris, Challamel, 1911. 8vo. 8+336 pp.
- ENSSLIN (M.). Elastizitätslehre für Ingenieure. I: Grundlagen und Allgemeines. (Sammlung Göschen, 519.) Leipzig, Göschen, 1911. 8vo. 140 pp. Cloth. M. 0.80
- FERGUSON (O. J.). Elements of electrical transmission. New York, Macmillan, 1911.
- FINGER (J.). Elemente der reinen Mechanik als Vorstudium für die analytische und angewandte Mechanik. 3te, neu bearbeitete und vermehrte Auflage. Wien, Hölder, 1911. 8vo. 15+842 pp. M. 24.50
- FÖPPL (A.). Vorlesungen über technische Mechanik. In 6 Bänden. 1ter Band: Einführung in die Mechanik. 4te Auflage. Leipzig, Teubner, 1911. 8vo. 15+424 pp. Cloth. M. 10.00
- GABRIEL (E.). Mécanique théorique et pratique. Tome 2: Statique pratique, dynamique théorique et pratique. Paris, Poussielgue, 1911. 8vo. 15+393 pp.
- GLEICHEN (A.). Die Theorie der modernen optischen Instrumente. Stuttgart, Enke, 1911. 8vo. 12+332 pp. Cloth. M. 12.00
- GRAETZ (L.). L'électricité et ses applications. Traduit sur la 15e édition allemande par G. Tardy. Paris, Masson, 1911. 8vo. 20+640 pp. Fr. 12.00
- GUILLOT (L.). Cours de mécanique rédigé conformément aux nouveaux programmes des Ecoles nationales d'arts et métiers. Tome 1er: Principes et théorèmes généraux de la mécanique. Statique graphique. Résistance des matériaux. Paris, Béranger, 1911. 8vo. 432 pp.
- HINCKLEY (F. C.) and RAMSAY (W. W.). The slide rule. Boston, Hinkley, 1911. 16mo. 104 pp. \$1.75
- HORN (F.). Die dynamischen Wirkungen der Wellenbewegung auf die Langebeanspruchung des Schiffskorpers. Berlin, Springer, 1910. 8vo. 118 pp. M. 3.00
- HUGHES (H. J.) and SAFFORD (A. T.). A treatise on hydraulics. New York, Macmillan, 1911.
- JONES (H. C.). Electrical nature of matter and radioactivity. 2nd edition, completely revised. New York, Van Nostrand, 1910. 8vo. 9+210 pp. \$2.00
- JOURDAIN (P. E. B.). See MACH (E.).
- LESSINE (L.) et SURET (L.). Introduction mathématique à l'étude de l'économie politique. Paris, Alcan, 1911. 16mo. 6+190 pp. Fr. 3.00
- MACH (E.). History and the root of the principles of conservation of energy. Translation by P. E. B. Jourdain. Chicago, Open Court Pub. Co., 1911. \$1.25

- MARCOLONGO (R.). Theoretische Mechanik. Deutsche Bearbeitung von H. E. Timerding. 1ter Band: Kinematik und Statik. Leipzig, Teubner, 1911. 8vo. 8+346 pp. Cloth. M. 11.00
- MATRICULATION mechanics papers. London University, from 1898 to 1900 and from September, 1902, to January, 1911. London, Clive, 1911. 8vo. 96 pp. Sewed. 1s. 6d.
- MOLDENHAUER (P.). Das Versicherungswesen. I: Allgemeine Versicherungslehre. (Sammlung Goschen, 262.) Neue Auflage. Leipzig, Göschen, 1911. 8vo. 158 pp. Cloth. M. 0.80
- PARKER (G. W.). Elements of mechanics. For the use of schools and colleges. London, Longmans, 1911. 8vo. 258 pp. 4s. 6d.
- POMINI (O.). Costruzione di macchine. I: Elasticità e resistenza dei materiali. (Biblioteca tecnica.) Milano, Hoepli, 1911. 8vo. 19 +509 pp. L. 9.00
- RAMSAY (W. W.). See HINCKLEY (F. C.).
- ROHR (M. v.). Die optischen Instrumente. (Aus Natur und Geisteswelt, 88.) 2te, vermehrte und verbesserte Auflage. Leipzig, Teubner, 1911. 8vo. 6+140 pp. Cloth. \$1.25
- RÖTHLISBERGER (J.). Moments sur les appuis des poutres continues dont le moment d'inertie est constant et dont les travées intermédiaires ont la même portée. Torino, Unione tipografico-editrice, 1911. 8vo. 7+142 pp. L. 5.00
- ROUBAUDI (C.). Cours de géométrie descriptive pour l'enseignement secondaire. 6e édition, revue, conforme au programme du 27 Juillet 1905. Fascicule 1er: Second cycle, classe de 1re (sections C et D). Paris, Masson, 1911. 8vo. 7+157 pp. Fr. 2.50
- SAFFORD (A. T.). See HUGHES (H. J.).
- SCHÖLER (R.). Einführung in den Brückenbau. Leipzig, Voigt, 1911. 8vo. 8+109 pp. M. 6.00
- . Hilfsbuch für Hochbautechniker, enthaltend eine Sammlung mathematisch-technischer Tabellen. Leipzig, Voigt, 1911. 8vo. 8+122 pp. Cloth. M. 3.00
- . Hilfsbuch für Tiefbautechniker, enthaltend eine Sammlung mathematisch-technischer Tabellen. Leipzig, Voigt, 1911. 8vo. 8+145 pp. Cloth. M. 3.00
- SMOLEY (C.). Tables of logarithms, squares, trigonometric functions, etc. For engineers, architects, and students. 6th edition, revised and enlarged. New York, Engineering News, 1911. 12mo. 329+171 pp. Leather. \$3.50
- STAUS (A.). Der Indikator und seine Hilfseinrichtungen. Berlin, Springer, 1911. 8vo. 7+188 pp. Cloth. M. 6.00
- SURET (L.). See LESEINE (L.).
- TAYLOR (T. U.). Backbone of perspective. Chicago, M. C. Clark, 1910. 12mo. 56 pp. \$1.00
- THURSTON (A. P.). Elementary aeronautics, or the science and practice of aerial machines. London, Whittaker, 1911. 8vo. 134 pp. 3s. 6d.
- TIMERDING (H. E.). See MARCOLONGO (R.).

VIEWEGGER (H.). Recueil de problèmes avec solutions sur l'électricité et ses applications pratiques. Traduction française de G. Capart. 2e édition, revue, corrigée et augmentée de nouveaux chapitres. Paris, Dunod et Pinat, 1911. 8vo. 16+403 pp. Fr. 9.00

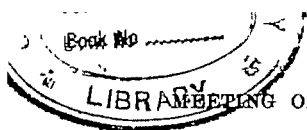
WEBER (H.) und WELLSTEIN (J.). Encyklopadie der Elementarmathematik. 3ter Band: Angewandte Elementarmathematik. 1ter Teil: Mathematische Physik. Bearbeitet von R. H. Weber. 2te Auflage. Leipzig, Teubner, 1910. 8vo. 12+536 pp. Cloth. M. 12.00

• WEITBRECHT. Lehrbuch der Vermessungskunde. 2ter (Schluss-) Teil: Vertikalmessungen. Stuttgart, Wittwer, 1911. 8vo. 6+306 pp. Cloth. M. 8.50

WELLSTEIN (J.). See WEBER (H.).

WITTENBAUER (F.). Aufgaben aus der technischen Mechanik. 1ter Band: Allgemeiner Teil. 773 Aufgaben nebst Lösungen. 2te, verbesserte Auflage. Berlin, Springer, 1911. 8vo. 11+301 pp. Cloth. M. 5.80

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THE APRIL MEETING OF THE SAN FRANCISCO SECTION.

THE nineteenth regular meeting of the San Francisco Section of the American Mathematical Society was held at Stanford University on Saturday, April 8, 1911. The following members were present:

Professor R. E. Allardice, Professor H. F. Blichfeldt, Professor R. L. Green, Professor Charles Haseman, Professor M. W. Haskell, Professor L. M. Hoskins, Mr. C. Kuschke, Professor D. N. Lehmer, Professor J. H. McDonald, Mr. G. F. McEwen, Professor H. C. Moreno, Professor R. E. Moritz, Professor C. A. Noble, Professor E. W. Ponzer, Professor S. D. Townley.

Professor Lehmer occupied the chair. The following papers were read at this meeting:

(1) Professor R. E. ALLARDICE: "Note on the envelope of the directrices of a certain system of conics."

(2) Professor H. F. BLICHFELDT: "Linear homogeneous groups having irreducible invariant subgroups."

(3) Mr. A. F. CARPENTER: "Geometrical interpretations of quotientiation and its inverse."

(4) Professor W. F. DURAND: "A mechanism which solves certain differential equations."

(5) Professor M. W. HASKELL: "Note on the del Pezzo quintic."

(6) Mr. C. KUSCHKE: "The equations of the 10th degree, irreducible in a given rational domain, whose groups are transitive with two as well as five systems of imprimitivity" (preliminary communication).

(7) Professor D. N. LEHMER: "Certain theorems in line geometry."

(8) Mr. W. V. LOVITT: "Transformations of partial differential equations."

(9) Professor J. H. McDONALD: "The transformation of elliptic integrals."

(10) Professor R. E. MORITZ: "On the cubes of determinants of the second, third, and higher orders."

(11) Professor E. W. PONZER: "The calculus in technical literature."

(12) Mr. L. L. SMAIL: "Toward a function theory of a certain hypercomplex variable in two units."

Professor Durand was introduced by Professor Allardice, and Mr. Lovitt and Mr. Smail by Professor Moritz. Professor Durand's paper was read by Professor Moreno and the papers of Mr. Carpenter, Mr. Lovitt, and Mr. Smail by Professor Moritz. Abstracts of the papers are given below.

1. Professor Allardice shows how the relation connecting the six distances between four points in a plane, in the form given by Cayley, may be applied almost directly to find the envelope of the directrices of a system of similar conics through three points. The envelope is a curve of the fourth class.

2. Let G' be a transitive (irreducible) collineation group of order N' in n variables, contained invariantly in a group G of order $N = N' \cdot M$ in the same variables. The more important results stated in Professor Blichfeldt's paper are: 1) No prime number greater than $n + 1$ divides M ; 2) Any prime factor of M which is greater than $\frac{1}{2}(n + 1)$ must be a factor of N' or must be of the form $2^n \pm 1$.

3. In Mr. Carpenter's paper the quotiential coefficient

$$Q_x y = \log_{g_x} g y = \lim_{h \rightarrow 1} \log_h \frac{f(x+h)}{f(x)},$$

where $y = f(x)$ is first exhibited as measuring the relative rate of change of functionally connected areas, the interpretation being made in either rectangular or polar coordinates. Two loci are obtained, either one of which, when associated with $y = f(x)$, exhibits $Q_x y$ as the ratio of two lines. By a slight modification in the cartesian system, $Q_x y$ will measure the slope of $y = f(x)$. A locus is found such that the area under it is given by the process inverse to $Q_x y$, i. e., $\Pi e^{y(gx-1)}$.

4. The fundamental principle utilized by Professor Durand is that a wheel with a sharp edge rolling on a paper surface will always tend to move in its own plane. Hence, if provision is made for adjusting the inclination of this plane (always vertical) to a line in the horizontal plane taken as the axis of X , such that $a \cdot \tan \phi = f(x)$, it will result that the wheel will, in rolling, trace out the function determined by the differential

equation $a \cdot dy/dx = f(x)$. Again, if a second plane is provided upon which rolls a second sharp edged wheel representing d^2y/dx^2 and if this second plane is given a movement parallel to y and equal in amount to the y component of the movement of the wheel representing dy/dx , then it results that the various parts of the mechanism may be so connected that a tracing point will move in fulfillment of the condition

$$a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + y = f(x).$$

Special cases may be handled where one or more terms may disappear, one or more may become negative, or $f(x)$ become constant.

5. In this note Professor Haskell shows that when the cusps of a quintic with five real cusps are given, the cuspidal tangents can be found by a linear construction.

6. In this preliminary communication Mr. Kuschke presents the results thus far obtained in investigating the irreducible equations of the 10th degree. He derives the necessary and sufficient conditions for all possible types of abelian equations. He discusses in detail the cases

$$1) \quad x^{10} - c = 0$$

and

$$2) \quad x^{10} - 10c^2\rho_3^2\rho_4x^6 - 25c^3\rho_3^2\rho_4^2x^4 + (15\rho_3^2\rho_4^2 - 10\rho_3\rho_4^7)c^4x^2 - c = 0,$$

where

$$\rho_3 = \sqrt[5]{-\frac{e^6}{2} - \frac{e^2}{2c} \sqrt{e^6c^3 + 4e}},$$

$$\rho_4 = \sqrt[5]{-\frac{1}{c^3} + \frac{e^6}{2} + \frac{e^2}{2c} \sqrt{e^6c^3 + 4e}}, \text{ and } e \neq 0.$$

7. Professor Lehmer defines a line as at right angles to a flat pencil of rays if it is at right angles to the ray of the pencil which it intersects. He finds the locus of lines at right angles to a given line and to a given pencil to be a ruled cubic surface. He finds also the number of lines at right angles to two given pencils and meeting a given line. He determines the locus of lines at right angles to three given pencils; also

the number of lines at right angles to four given pencils. This last problem is equivalent to finding the number of lines in space such that the common perpendiculars between them and four fixed right lines meet the four fixed lines in given points.

8. Mr. Lovitt uses the infinitesimal transformation

$$X = x + \xi(x, y, z)\delta t, \quad Y = y + \eta(x, y, z)\delta t, \quad Z = z + \zeta(x, y, z)\delta t.$$

Formulas are found giving the effect of this transformation upon the partial derivatives. Recursion formulas are found and also two expressions for each partial derivative provided it is taken at least once with respect to both x and y . This furnishes a useful check. Explicit expressions are given for the changes in the first and second partial derivatives and also numerous partial differential equations of the second order which remain invariant under given transformations. The most general transformation is found which converts a linear partial differential equation in three variables, with variable coefficients, into another equation of the same form.

9. Professor McDonald's paper shows how to construct the involutions on which depend the transformations of elliptic integrals. Clifford discovered that these are furnished by the multiplication formulas. The consideration of the polygonal line inscribed and circumscribed to two conics gives, in the case of closure, an algebraic solution to the problem of Legendre and Jacobi.

10. Professor Moritz gives special forms for the cubes of determinants of the second, third, and higher orders, and demonstrates

$$\Delta_n^3 = \frac{1}{(c_1 d_1 \dots n_n)(c_2 d_2 \dots n_n)(c_3 d_3 \dots n_n)} \begin{vmatrix} A_1^2 & A_2^2 & A_3^2 \\ A_1 B_1 & A_2 B_2 & A_3 B_3 \\ B_1^2 & B_2^2 & B_3^2 \end{vmatrix},$$

where $\Delta_n = (a_1 b_2 c_3 \dots n_n)$ and where the large letters are the co-factors of the small letters in this determinant.

11. In a previous paper (*Science*, October 22, 1909) Professor Ponzer gave the results of an investigation into the various principles and applications of the calculus used in under-

graduate courses and the relative frequency of their occurrence. In the present paper a similar study is made of the calculus used by the practicing engineers, the more recent volumes of leading technical journals furnishing the data used. He discusses also the emphasis placed on calculus and the attitude of the engineers of different nationalities toward it.

12. Mr. Smail considers a hypercomplex number system in two units e_1, e_2 , where $e_1^2 = e_1$, $e_1 e_2 = e_2 e_1 = 0$, $e_2^2 = e_2$. He considers the elementary conceptions and the functionality of the simpler algebraic expressions. Differentiation is defined and its limitations pointed out; integration is defined and the integral of a hypercomplex variable expressed in terms of ordinary curvilinear integrals. He finds also the conditions that make the integral independent of the path of integration.

H. C. MORENO,
Secretary of the Section.

INVARIANT CONDITIONS THAT A p -ARY FORM MAY HAVE MULTIPLE LINEAR FACTORS.

BY PROFESSOR O. E. GLENN.

(Read before the American Mathematical Society, October 29, 1910.)

§ 1. *Introduction.*

IN the case of a ternary quadratic form the vanishing of the ordinary discriminant is the necessary and sufficient condition that the form be factorable into linear factors, distinct term for term. In the case of a cubic ternary form we can obtain in various ways a set of three independent rational, integral, and homogeneous functions of the coefficients, whose simultaneous vanishing furnishes the necessary and sufficient conditions that the form be factorable into distinct linear factors. The general theorem underlying these facts has been proved by Junker,* and is that the necessary and sufficient conditions that a p -ary form of order m be a product of m distinct linear factors consist in the simultaneous vanishing of a set of $\binom{m+p-1}{m} - m(p-1) - 1$ independent seminvariant† relations among the form's coefficients.

*Junker, "Die symmetrischen Funktionen u. s. w.," *Math. Annalen*, vol. 45

†Junker, "Ueber die Diff.-Gleichungen der Invarianten," loc. cit., vol. 64.

Several papers* have been published on the problem of determining such sets of conditions. Brill † and Gordan ‡ have each treated the problem by the symbolical methods, obtaining in each case a redundant set of conditions furnished by the identical vanishing of a covariant. Junker confined his attention to the problem of finding in terms of the coefficients the $\binom{m+p-1}{m} - 1$ identical relations which exist among the elementary symmetric functions of the m groups of p homogeneous variables of the form. His sets of relations also do not contain the minimum number. There exist $m(p-1)$ syzygies among them.

Not much attention seems to have been given however to the problem of determining a minimum set of conditions that a p -ary form f_{p^m} ($p > 2$) may possess *multiple* linear factors. It is the purpose of this note to show how a certain factor theorem for f_{3^m} , which we give in the first part of § 2, may be made the basis of various methods of determining such sets. In § 2 we determine conditions in order that a p -ary m -ic f_{p^m} should be a perfect m th power. In § 3 sets of conditions that f_{3^m} should involve a repeated factor are developed. Proof that the sets of conditions determined in the various cases really contain the minimum number is given in § 4.

§ 2. Conditions that a p -ary Form may be an m th Power.

Let

$$(1) f_{3^m} = x_2^m \phi_{0 \ x_1/x_2} + x_2^{m-1} \phi_{1 \ x_1/x_2} x_3 + x_2^{m-2} \phi_{2 \ x_1/x_2} x_3^2 + \dots + \phi_{m \ x_1/x_2} x_3^m,$$

where

$$x_2^{m-r} \phi_{r \ x_1/x_2} \equiv a_{m-r \ 0} x_1^{m-r} + a_{m-r-1 \ 1} x_1^{m-r-1} x_2 + a_{m-r-2 \ 2} x_1^{m-r-2} x_2^2 + \dots + a_{0 \ m-r} x_2^{m-r},$$

be a ternary form equal to the product of m linear factors. Then the (x_1, x_2) terms of those factors must be furnished by the linear factors of the binary form $x_2^m \phi_{0 \ x_1/x_2}$. Let the factors of the latter be assumed to be known and let them be

$$x_1 + r_i x_2 \quad (i = 1, 2, \dots, t),$$

* Brioschi, "Sulla condizioni per la decomposizione di una forma cubica ternaria, etc." *Annali di Matematica*, ser. 12, vol. 7.

† Brill, "Ueber die Zerfällung der Ternärformen in lineare Factoren," *Jahresbericht der Deutschen Math.-Vereinigung*, 1896, and *Math. Annalen*, vol. 50.

‡ Gordan, "Das Zerfallen der Kurven in gerade Linien," loc. cit., vol. 45.

where $x_1 + r_i x_2$ is of multiplicity α_i . We have proved in another paper* the following

THEOREM: *The ternary form f_{3m} can be factored into factors of the respective orders $\alpha_1, \alpha_2, \dots, \alpha_t$, which are rational and integral in the coefficients of the form f_{3m} itself on the one hand, and in the quantities r_i on the other, linear in the coefficients; according to the formula*

$$(2) \quad f_{3m} = \prod_{i=1}^t \left[\frac{\partial^{\alpha_i} \phi_{0-r_i}}{\partial r_i^{\alpha_i}} x_i^{\alpha_i} - \alpha_i \frac{\partial^{\alpha_i-1} \phi_{1-r_i}}{\partial r_i^{\alpha_i-1}} x_i^{\alpha_i-1} y_i + \alpha_i (\alpha_i - 1) \right. \\ \left. \times \frac{\partial^{\alpha_i-2} \phi_{2-r_i}}{\partial r_i^{\alpha_i-2}} x_i^{\alpha_i-2} y_i^2 - \dots + (-1)^{\alpha_i} \alpha_i \phi_{\alpha_i-r_i} y_i^{\alpha_i} \right]$$

$$x_1 + r_i x_2 / x_3 = x / y_i \quad (i = 1, 2, \dots, t).$$

If the Hessian covariant of one of these factors of f_{3m} , considered as a binary form in (x, y) , vanishes identically, f_{3m} has a linear factor of multiplicity α_i , and conversely. Thus if $m = 2$,

$$f_{32} = ax_1^2 + 2hx_1x_2 + bx_2^2 + 2gx_1x_3 + 2fx_2x_3 + cx_3^2,$$

we get in this manner necessary and sufficient conditions that the *general* ternary quadratic be a perfect square. These are †

$$C_1 = af^2 - 2fgh + bg^2 = 0,$$

$$C_2 = ab - h^2 = 0, \quad C_3 = ac - g^2 = 0.$$

They are independent, that is, as we shall show in general in § 4, they form a minimum set of conditions that a *general* form f_{32} be a perfect square.

We proceed now to generalize the results in the last paragraph, deriving necessary and sufficient conditions that the factorable p -ary form f_{pm} be a perfect m th power.

A notation can be adopted for f_{pm} which exhibits the form as a sum of $p - 2$ ternary forms plus a residue Ψ_{pm} which is not in the shape of a ternary form. We use for the ternary summands a notation like (1), e. g.,

* BULLETIN, vol. 17, No. 2, p. 63.

† Cf. Bromwich, Quadratic Forms, p. 9.

where $x/y = (x_1 + r_1 x_2)/x_{j+2}$, a constant factor being neglected on the right.

In order that $f_{p,m}$ should be a perfect m th power, $x_2^m \phi_0 x_1/x_2$ and the $p-2$ binary forms $X^{(j)}$ ($j=1, 2, \dots, p-2$) must be m th powers. The vanishing of $H\phi_0$ and the $p-2$ Hessian covariants of $X^{(j)}$ ($j=1, 2, \dots, p-2$) are necessary and sufficient conditions for this.

The Hessian of $X^{(j)}$ is

$$K_j = \begin{vmatrix} X_{xx}^{(j)} & X_{xy}^{(j)} \\ X_{yx}^{(j)} & X_{yy}^{(j)} \end{vmatrix},$$

and we have

$$X_{xx}^{(j)} = \left[m \sum_{h=0}^{m-2} (m-h)(m-h-1) a_{m-h-0 \dots 0 \overset{j}{\dots} \overset{p-j-2}{h} 0 \dots 0} x^{m-h-2} y^h, \right.$$

$$X_{xy}^{(j)} = \left[m \sum_{h=0}^{m-2} (m-h-1)(h+1) a_{m-h-1 \dots 0 \overset{j}{\dots} \overset{p-j-2}{h+1} 0 \dots 0} x^{m-h-2} y^h, \right.$$

$$X_{yy}^{(j)} = \left[m \sum_{h=0}^{m-2} (h+1)(h+2) a_{m-h-2 \overset{j}{\dots} 0 \overset{p-j-2}{h+2} 0 \dots 0} x^{m-h-2} y^h, \right.$$

formulas which give explicitly, in terms of the coefficients of $f_{p,m}$, the values of the determinants K_j .

Since the $p-1$ binary forms $x_2^m \phi_0 x_1/x_2$, $X^{(j)}$ ($j=1, 2, \dots, p-2$) are all independent, the $p-1$ Hessians are all independent. Thus a set of necessary and sufficient conditions that a p -ary form $f_{p,m}$ which is linearly factorable, should be the m th power of a linear form consist in the simultaneous vanishing of the $p-1$ binary Hessian covariants $H\phi_0$, K_j ($j=1, 2, \dots, p-2$).

It will be proved in § 4 that when the $\binom{m+p-1}{p-1} - m(p-1) - 1$ conditions that the general p -ary mic be linearly factorable are joined to the above $p-1$ Hessians we get a minimum set of conditions that $f_{p,m}$ (in general) be an m th power.

§ 3. Conditions that $f_{p,m}$ Contain a Squared Factor.

If we put $y=1$, $m=3$, $\alpha_1=\alpha_2=\alpha_3=1$ in (2), carry out the indicated multiplications and divide by the coefficient of x^3 , we get

$$\begin{aligned}
 f_{33} = & \prod_{i=1}^3 \left[x_1 + r_i x_2 - \frac{\phi_{1-r_1}}{\phi'_{0-r_1}} \right] + \left(a_{10} - \sum \frac{\phi_{1-r_1} \phi_{1-r_2}}{\phi'_{0-r_1} \phi'_{0-r_2}} \right) x_1 \\
 (5) \quad & + \left(a_{01} - \sum \frac{\phi_{1-r_1} \phi_{1-r_2}}{\phi'_{0-r_1} \phi'_{0-r_2}} r_3 \right) x_2 + \left(a_{00} + \frac{\phi_{1-r_1} \phi_{1-r_2} \phi_{1-r_3}}{\phi'_{0-r_1} \phi'_{0-r_2} \phi'_{0-r_3}} \right),
 \end{aligned}$$

where the summations are to be taken with reference to the r_i , being symmetric in the r_i . Since f_{33} is factorable by hypothesis, the quantities in the parentheses are equal to zero. In fact these expressions equated to zero furnish a set of necessary and sufficient conditions that the *general* f_{33} be factorable into linear factors distinct term for term.

If we write

$$\begin{aligned}
 \Delta_1 &= 3a_{30} \frac{\partial}{\partial a_{20}} + 2a_{21} \frac{\partial}{\partial a_{11}} + a_{12} \frac{\partial}{\partial a_{03}}, \\
 \Delta_2 &= 3a_{03} \frac{\partial}{\partial a_{02}} + 2a_{12} \frac{\partial}{\partial a_{11}} + a_{21} \frac{\partial}{\partial a_{20}},
 \end{aligned}$$

and let D be the discriminant of $x_2^3 \phi_{0 \ x_1/x_2}$ and R_3 the resultant of the latter taken with $x_2^2 \phi_{1 \ x_1/x_2}$, these three conditional relations become

$$\begin{aligned}
 H_3 &= a_{00} D - R_3 = 0, \\
 (6) \quad H_2 &= a_{01} D - \Delta_2 R_3 = 0, \\
 H_1 &= a_{10} D + \Delta_1 R_3 = 0.
 \end{aligned}$$

If a ternary form f_{33} contains a squared factor it is, of course, linearly factorable as a whole. Then (2) gives

$$\begin{aligned}
 f_{33} = & \left[x_1 + r_1 x_2 - \frac{\phi'_{1-r_1}}{\phi''_{0-r_1}} \right]^2 \left[x_1 + r_2 x_2 - \frac{\phi_{1-r_2}}{\phi'_{0-r_2}} \right] \\
 & + \left(a_{10} - 2 \frac{\phi'_{1-r_1} \phi_{1-r_2}}{\phi''_{0-r_1} \phi'_{0-r_2}} - \frac{\phi_{1-r_1}^2}{\phi''_{0-r_1}^2} \right) x_1 \\
 & + \left(a_{01} - 2 \frac{\phi'_{1-r_1} \phi_{1-r_2}}{\phi''_{0-r_1} \phi'_{0-r_2}} r_1 - \frac{\phi_{1-r_1}^2}{\phi''_{0-r_1}^2} r_2 \right) x_2 + \left(a_{00} - \frac{\phi_{1-r_1}^2 \phi_{1-r_2}}{\phi''_{0-r_1}^2 \phi'_{0-r_2}} \right).
 \end{aligned}$$

Evaluating the expressions in the parentheses we get a minimum set of conditions that the *general* f_{33} contain a squared factor, e. g.,

$$\begin{aligned}
 D &= 0, \quad R_3 = 0, \\
 \Xi_1 &= a_{10} \frac{\partial D}{\partial a_{3-ii}} + \Delta_1 \frac{\partial R_3}{\partial a_{3-ii}} = 0, \\
 \Xi_2 &= a_{01} \frac{\partial D}{\partial a_{3-ii}} - \Delta_2 \frac{\partial R_3}{\partial a_{3-ii}} = 0, \\
 \Xi_3 &= a_{00} \frac{\partial D}{\partial a_{3-ii}} - \frac{\partial R_3}{\partial a_{3-ii}} = 0,
 \end{aligned}
 \tag{7}$$

(i = any number of the set 0, 1, 2, 3).

In § 2 we have already developed a set of conditions that f_{33} be a perfect cube, assuming it to be linearly factorable. The method of this section gives us a minimum set, without assumption. For if the resultant and the first subresultant of the first partial derivatives of $x_2^3 \phi_{0x_1/x_2}$ vanish, say $D = 0$, $D_1 = 0$, and also the resultant and the first subresultant of $x_2^3 \phi_{0x_1/x_2}$ and $x_2^3 \phi_{1x_1/x_2}$, say $R_3 = 0$, $S_3 = 0$, then (2) gives

$$\begin{aligned}
 f_{33} &= \left[\dot{x}_1 + r_1 x_2 - \frac{\phi_{1-r_1}''}{\phi_{0-r_1}'''} \right]^3 + \left(a_{10} - 3 \frac{\phi_{1-r_1}''^2}{\phi_{0-r_1}'''} \right) x_1 \\
 &\quad + \left(a_{01} - 3r_1 \frac{\phi_{1-r_1}''^2}{\phi_{0-r_1}'''} \right) x_2 + \left(a_{00} + \frac{\phi_{1-r_1}''^3}{\phi_{0-r_1}'''} \right),
 \end{aligned}$$

from which we get the necessary and sufficient conditions that the *general* ternary cubic be a cube, in the form

$$\begin{aligned}
 D &= 0, \quad D_1 = 0, \quad R_3 = 0, \quad S_3 = 0, \\
 \Gamma_1 &= a_{10} \frac{\partial^2 D}{\partial a_{3-ii}^2} + \Delta_1 \frac{\partial^2 R_3}{\partial a_{3-ii}^2} = 0, \\
 \Gamma_2 &= a_{01} \frac{\partial^2 D}{\partial a_{3-ii}^2} - \Delta_2 \frac{\partial^2 R_3}{\partial a_{3-ii}^2} = 0, \\
 \Gamma_3 &= a_{00} \frac{\partial^2 D}{\partial a_{3-ii}^2} - \frac{\partial^2 R_3}{\partial a_{3-ii}^2} = 0
 \end{aligned}
 \tag{8}$$

(i = any number of the set 0, 1, 2, 3).

We will now show still another method of applying the

theorem of § 2 in the determination of conditions for multiple factors of a ternary form f_{3m} .

Consider the case where f_{3m} has a single linear factor of multiplicity 2 and let it have, besides, $m-2$ distinct linear factors. Then in (2) $\alpha_1 = 2$, $\alpha_2 = \alpha_3 = \dots = 1$, and

$$\nabla_{r_1} = \phi_{1-r_1}^2 - 2\phi_{0-r_1}''\phi_{2-r_1} = 0.$$

Hence f_{3m} will have squared factors (and none of multiplicity higher than 2) when and only when there exist roots r , common to the three equations

$$\phi_{0-r} = 0, \quad \phi_{0-r}' = 0, \quad \nabla_r = 0,$$

provided the resultant R of ϕ_{0-r} and ϕ_{0-r}'' does not vanish. Thus if D is the discriminant of ϕ_{0-r} , and R_1 is the resultant of ∇_r and the greatest common divisor of ϕ_{0-r} and ϕ_{0-r}' , a set of necessary and sufficient conditions that a linearly factorable f_{3m} should contain linear factors to the second power is

$$(9) \quad D = 0, \quad R \neq 0, \quad R_1 = 0.$$

§ 4. Proof that the Conditions Give Minimum Sets.

We can prove that the $\binom{m+p-1}{p-1} - m(p-1) - 1$ factorability conditions for any general f_{pm} together with the $p-1$ Hessian conditions $H\phi_0 = 0$, $K_j = 0$ ($j = 1, 2, \dots, p-2$) of § 2 furnish a minimum set of invariant conditions in order that the general f_{pm} be an m th power.

The number of identical relations which exist among the elementary symmetric functions of m groups of p homogeneous variables is $\binom{m+p-1}{p-1} - 1$. If these variables are the mp coefficients of m linear factors of f_{pm} , the elementary symmetric functions are the coefficients of that form. If the factors are distinct, term for term, the identical relations, considered as relations among the coefficients of the form, will be $\binom{m+p-1}{p-1} - 1$ equations in $m(p-1)$ non-homogeneous variables, so that only

$$\binom{m+p-1}{p-1} - m(p-1) - 1$$

of the relations are independent. An independent set forms a minimum set of conditions that f_{pm} be factorable into m linear factors, distinct term for term. Now if f_{pm} is an m th power, the number of variables is only $p-1$. Hence the minimum

number of conditions in order that f_{pm} may be an m th power is $\binom{m+p-1}{m} - p$.

Each one of the Hessians $H\phi_0, K_j$ ($j = 1, 2, \dots, p-2$) is of order $2m-4$ in the variables which it contains, and so the number of vanishing coefficients in each is $2m-3$. Hence these give $(2m-3)(p-1)$ conditions in addition to the $\binom{m+p-1}{m} - m(p-1) - 1$ assumed ones. But of the $2m-3$ conditions obtained by equating to zero the coefficients of a binary Hessian covariant only $m-1$ are independent, as the m coefficients of the form can all be expressed in terms of a single quantity when the Hessian vanishes. Hence we have as a total number of conditions given by the original factorability conditions of f_{pm} and the Hessians

$$\binom{m+p-1}{m} - m(p-1) - 1 + (m-1)(p-1) = \binom{m+p-1}{m} - p,$$

which is thus the minimum number required. Hence the relations derived in § 2 furnish a minimum set.

In the same way it may be shown that (6), (7), (8), (9) are all minimum sets.

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THE GENERAL TERM OF A RECURRING SERIES.

BY PROFESSOR ARTHUR RANUM.

(Read before the San Francisco Section of the American Mathematical Society, September 26, 1908.)

1. The principal theorem of this note expresses the general term of a recurring series rationally in terms of the first few terms and the constants of the scale of relation. Although I derived it in 1908, I have only recently learned that practically the same theorem was published by D'Ocagne in 1894 (*Journal de L'Ecole Polytechnique*, volume 64, pages 151-224) and by Netto in 1895 (*Monatshefte für Mathematik und Physik*, volume 6, pages 285-290). Nevertheless it may be worth while to publish my own work for three reasons: first, because my proof is simpler than those of D'Ocagne and Netto; second, because I have stated the result in a more explicit form than that of either of these authors*; third, because I have applied

* D'Ocagne gives an explicit statement of the theorem (p. 163) for the special case in which the series is a "suite fondamentale," but not for the general case.

the result to the series of powers of a matrix, and this application is, I believe, entirely new.*

2. Let $U = u_0 + u_1 + \dots + u_{n-1} + \dots + u_m + \dots$ be any recurring series of order n , and let

$$(1) \quad u_m = a_1 u_{m-1} + a_2 u_{m-2} + \dots + a_n u_{m-n} \quad (m=n, n+1, \dots)$$

be its scale of relation. The general term u_m is evidently a linear homogeneous function of the first n terms $u_0, \dots, u_{n-1}$ and a rational integral function of the n constants $a_1, \dots, a_n$ of the scale of relation. Our problem is to determine the explicit form of this function.

The corresponding power series will be

$$(2) \quad U(x) = u_0 + u_1 x + \dots + u_{n-1} x^{n-1} + U_n(x),$$

where

$$(3) \quad U_n(x) = u_n x^n + \dots + u_{2n-1} x^{2n-1} + \dots;$$

that is, $U_n(x)$ is obtained from $U(x)$ by removing the first n terms. These series will always be convergent for a certain range of values of x . In all that follows we assume that x is chosen within that range.

From (1) and (3) we easily derive the identity

$$\begin{aligned} (1 - a_1 x - a_2 x^2 - \dots - a_n x^n) \cdot U_n(x) &= u_n x^n \\ &+ (u_{n+1} - a_1 u_n) x^{n+1} + (u_{n+2} - a_1 u_{n+1} - a_2 u_n) x^{n+2} \\ &+ \dots + (u_{2n-1} - a_1 u_{2n-2} - \dots - a_{n-1} u_n) x^{2n-1} \\ &= (a_1 u_{n-1} + \dots + a_n u_0) x^n + (a_2 u_{n-1} + \dots + a_n u_1) x^{n+1} \\ &+ \dots + (a_{n-1} u_{n-1} + a_n u_{n-2}) x^{2n-2} + a_n u_{n-1} x^{2n-1}, \end{aligned}$$

which can be written in the form

$$(4) \quad U_n(x) = \frac{u'_0 x^n + u'_1 x^{n+1} + \dots + u'_{n-2} x^{2n-2} + u'_{n-1} x^{2n-1}}{1 - a_1 x - a_2 x^2 - \dots - a_n x^n},$$

provided we define the auxiliary quantities $u'_0, \dots, u'_{n-1}$ by the equations

* For other applications see the papers of D'Ocagne and Netto, especially the former.

for the general term u_{m+n} of the recurring series U , where the n consecutive coefficients $A_m, \dots, A_{m-n+1}$, defined by (6) and (8), are rational integral functions of the constants $a_1, \dots, a_n$ of the scale of relation, and where the auxiliary quantities $u'_0, \dots, u'_{n-1}$, defined by (5), are linear homogeneous functions of the first n terms $u_0, \dots, u_{n-1}$ of the series.

Application to Matrices.

3. Let us now apply the formula (9) so found to the recurring series that consists of the successive positive integral and zero powers of a linear homogeneous substitution in n variables, or in other words of an n -ary matrix $L = (l_{ij})$, where l_{ij} ($i, j = 1, \dots, n$) is the element in the i th row and j th column. Let L^0 be the corresponding unit matrix. We wish to express all the powers of L as linear homogeneous functions of the first n powers $L^0, L, \dots, L^{n-1}$.* The first $n+1$ powers of L satisfy the well-known † Hamilton-Cayley equation

$$(10) \quad L^n = a_1 L^{n-1} + a_2 L^{n-2} + \dots + a_{n-1} L + a_n L^0,$$

where

$$a_1 = \sum_{i=1}^n l_{ii}, \quad -a_2 = \sum_{i=1}^n \sum_{j=1}^n \begin{vmatrix} l_{ii} & l_{ij} \\ l_{ji} & l_{jj} \end{vmatrix} \quad (i < j), \dots,$$

and

$$(-1)^{n-1} a_n = |l_{ij}|,$$

the determinant of L . Multiplying (10) by L^{n-1} , we obtain the scale of relation

$$L^n = a_1 L^{n-1} + \dots + a_n L^{n-n}.$$

Hence we have only to define a set of auxiliary matrices $L_0, L_1, \dots, L_{n-1}$ by the equations

$$L_0 = a_1 L^{n-1} + \dots + a_n L^0,$$

$$L_1 = a_2 L^{n-1} + \dots + a_n L^1,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$L_{n-2} = a_{n-1} L^{n-1} + a_n L^{n-2},$$

$$L_{n-1} = a_n L^{n-1},$$

* This problem I stated and partly solved in the BULLETIN, vol. 13 (1907), pp. 337-338.

† Cf. Bôcher, Higher Algebra (1907), p. 296.

and our problem is completely solved by the equation

$$L^{m+n} = A_m L_0 + A_{m-1} L_1 + \cdots + A_{m-n+1} L_{n-1},$$

where the A 's are scalar quantities defined, as before, by (6) and (8).

To illustrate this method, we shall calculate the 12th power of the ternary matrix

$$L = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & 2 \end{pmatrix}.$$

In this case $a_1 = 2$, $a_2 = -2$, $a_3 = 1$,

$$L_0 = \begin{pmatrix} 1 & -2 & 2 \\ 2 & -3 & 2 \\ 2 & -2 & 1 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 0 & 1 & -2 \\ -2 & 4 & -3 \\ -3 & 4 & -2 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -2 & 2 \\ 2 & -3 & 2 \end{pmatrix},$$

$$L^{12} = A_9 L_0 + A_8 L_1 + A_7 L_2,$$

$$\begin{aligned} A_7 = & (a_1^7 + 6a_1^5 a_2 + 10a_1^3 a_2^2 + 4a_1 a_2^3) \\ & + (5a_1^4 + 12a_1^2 a_2 + 3a_2^2) + 3a_1 a_3^2 = 2; \end{aligned}$$

similarly $A_8 = 2$, $A_9 = 1$. Therefore

$$L^{12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

CORNELL UNIVERSITY,
January, 1911.

RELATIONS BETWEEN THE GRAMIAN, THE
WRONSKIAN, AND A THIRD DETERMI-
NANT CONNECTED WITH THE
PROBLEM OF LINEAR
DEPENDENCE.

BY PROFESSOR D. R. CURTISS.

(Read before the American Mathematical Society, April 28, 1911.)

THE three determinants here discussed, the Gramian G , the Wronskian W , and the third determinant D , are formed for n functions, real or complex, $f_1, f_2, \dots, f_n$, which we shall here suppose for simplicity to be functions of a real variable x , as follows:

$$G = \begin{vmatrix} \int_a^b f_1(x) \bar{f}_1(x) dx & \int_a^b f_1(x) \bar{f}_2(x) dx & \dots & \int_a^b f_1(x) \bar{f}_n(x) dx \\ \int_a^b f_2(x) \bar{f}_1(x) dx & \int_a^b f_2(x) \bar{f}_2(x) dx & \dots & \int_a^b f_2(x) \bar{f}_n(x) dx \\ \vdots & \vdots & \dots & \vdots \\ \int_a^b f_n(x) \bar{f}_1(x) dx & \int_a^b f_n(x) \bar{f}_2(x) dx & \dots & \int_a^b f_n(x) \bar{f}_n(x) dx \end{vmatrix},$$

where $\bar{f}(x)$ is the conjugate of $f(x)$;

$$W = \begin{vmatrix} f_1(x) & f_2(x) & \dots & f_n(x) \\ \frac{d}{dx} f_1(x) & \frac{d}{dx} f_2(x) & \dots & \frac{d}{dx} f_n(x) \\ \vdots & \vdots & \dots & \vdots \\ \frac{d^{n-1}}{dx^{n-1}} f_1(x) & \frac{d^{n-1}}{dx^{n-1}} f_2(x) & \dots & \frac{d^{n-1}}{dx^{n-1}} f_n(x) \end{vmatrix};$$

$$D = \begin{vmatrix} f_1(x_1) & f_2(x_1) & \dots & f_n(x_1) \\ f_1(x_2) & f_2(x_2) & \dots & f_n(x_2) \\ \vdots & \vdots & \dots & \vdots \\ f_1(x_n) & f_2(x_n) & \dots & f_n(x_n) \end{vmatrix}.$$

It is well known * that if the functions $f_1, f_2, \dots, f_n$ are continuous (or even more general conditions may be given), the vanishing of G is a necessary and sufficient condition for their linear dependence in the interval $[a, b]$. The identical vanishing of W is a necessary condition provided the derivatives involved exist, and with certain added hypotheses it is also sufficient.†

It has perhaps been previously noted, though I have been able to find no explicit reference on this point,‡ that the identical vanishing of D for all values of the independent variables $x_1, x_2, \dots, x_n$ in $[a, b]$ is a necessary and sufficient condition that $f_1(x), f_2(x), \dots, f_n(x)$ be linearly dependent in $[a, b]$, i. e., that there exist a linear relation valid at every point of $[a, b]$

$$(1) \quad c_1 f_1(x) + c_2 f_2(x) + \dots + c_n f_n(x) = 0,$$

where at least one of the constants c is not zero. That $D \equiv 0$ is a necessary condition is obvious; to see this, if the notation is so chosen that $c_n \neq 0$, we have only to multiply the i th column of D by c_i/c_n ($i = 1, 2, \dots, n-1$) and add to the last column, thus making each element of that column zero on account of (1). Conversely, if $D \equiv 0$, suppose the minor of the element $f_n(x_n)$, which we may denote by D_n , is not zero for at least one set of values of $x_1, x_2, \dots, x_{n-1}$ in $[a, b]$. Such a value being given to these variables we have a relation (1) if we replace x_n by x and expand D in terms of the elements of the last row and their minors. If $D_n \equiv 0$ in $[a, b]$, then either the same is true of $f_1(x)$, in which case (1) will hold if $c_1 \neq 0, c_2 = \dots = c_n = 0$, or else there is a determinant formed by suppressing the last m rows and the last m columns of D which vanishes identically in $[a, b]$ but in which the minor of the last element of the last row is different from zero for at least one set of values of the variables concerned in $[a, b]$. This determinant is of the same type as D , and arguing as before we establish a linear relation between $f_1, f_2, \dots, f_m$, which may be regarded as a relation (1) where $c_{m+1} = \dots = c_n = 0$.

* Cf. G. Kowalewski, Einführung in die Determinantentheorie, Chap. XV.; also Böcher's review of this chapter in the BULLETIN, vol. 17 (1910), p. 126.

† Cf. in addition to the above citations a paper by the author, "The vanishing of the Wronskian and the problem of linear dependence," *Mathematische Annalen*, vol. 65 (1908), p. 282, where further references are given.

‡ Possibly this is implied in a remark of the paper of Richardson and Hurwitz referred to on page 464.

If we compare the criteria for linear dependence furnished by the vanishing of these three determinants, we see that the condition $D \equiv 0$ applies in all cases where the functions are defined for every point of $[a, b]$, while $G = 0$ requires the continuity, or at least the integrability, of the functions concerned, and $W \equiv 0$ not only implies the existence of derivatives of order up to $n - 1$ but is even then insufficient without additional hypotheses. We may note also that the criteria $D \equiv 0$ and $G = 0$ apply for functions of more than one variable, with suitable change of notation, while no single identity of the same sort can replace $W \equiv 0$.

A certain interest attaches to formulas connecting G , D , and W . Thus Meder has recently proved the relation *

$$(2) \quad \left. \frac{d^{n-1} G}{db^{n-1}} \right|_{b=a} = M [W(a)]^2,$$

where M is a constant, and $W(a)$ indicates the Wronskian formed for $x = a$. This formula cannot, however, be reversed so as to express G in terms of W . We now proceed to obtain relations connecting D and G , and D and W , by combining two of which G may be expressed in terms of W .

An identity connecting D and G may be derived at once, in case the functions $f_1, f_2, \dots, f_n$ are all real, from the formula (4) given by Richardson and Hurwitz in a recent article in the BULLETIN.† If we put $k = 1$, this formula becomes, with appropriate changes in notation,

$$(3) \quad G = \frac{1}{n!} \int_a^b \dots \int_a^b D^2 dx_1 \dots dx_n.$$

It can easily be shown that if the functions concerned are complex the corresponding formula is

$$(3') \quad G = \frac{1}{n!} \int_a^b \dots \int_a^b D \bar{D} dx_1 \dots dx_n,$$

where $\bar{D}$ is the conjugate of D . These formulas put in evidence the property $G \geq 0$, and give a simple proof that $G = 0$

* "Ueber den Zusammenhang zwischen den Determinanten von Gram und Wronski," *Monatshefte für Mathematik und Physik*, vol. 21 (1910), p. 336.

† "Note on determinants whose terms are certain integrals," vol. 16 (1909), no 1, p. 14.

is a necessary and sufficient condition for linear dependence. They are valid, with suitable changes of notation, for functions of any number of variables.

To express W in terms of D , formulas (2) and (3) or (3') may be combined; or we may verify directly the result

$$(4) \quad W(x) = \left[\frac{\partial^{k^n(n-1)}}{\partial x_2 \partial x_3^2 \dots \partial x_n^{n-1}} D \right]_{x_1=x_2=\dots=x_n=x}.$$

The problem of finding a formula which will give D in terms of W in such a way as to put in evidence the vanishing of D when $W \equiv 0$ in $[a, b]$ is impossible of solution except under added hypotheses, since $W \equiv 0$ is not alone a sufficient condition for linear dependence. If, however, one of the $n-1$ rowed determinants of the matrix formed by the first $n-1$ rows of W vanishes at no point of $[a, b]$, for example the determinant W_n formed by suppressing the last column of this matrix, we may obtain a relation of the kind desired with the aid of the well-known formula of Frobenius for Wronskians.* If we use the notation $W_r(x)$ to indicate the determinant formed by suppressing the last row and the r th column of W , and $W_{r,s}(x)$ for the determinant formed by suppressing the last row and the s th column of W_r , a special case of Frobenius' formula may be written

$$(5) \quad W(x) \cdot W_{n,r}(x) = W_n(x) \frac{d}{dx} W_r(x) - W_r(x) \frac{d}{dx} W_n(x) \\ (r = 1, 2, \dots, n-1).$$

Since $W_n(x) \neq 0$ in $[a, b]$, we may divide (5) by $[W_n(x)]^2$ and integrate, thus obtaining

$$(6) \quad \int_a^x \frac{W(\xi) W_{n,r}(\xi)}{[W_n(\xi)]^2} d\xi = \frac{W_r(x)}{W_n(x)} - \frac{W_r(z)}{W_n(z)} \\ (r = 1, 2, \dots, n-1).$$

If we replace the last row of W by the first, so that the resulting determinant vanishes identically, we have, on expanding according to the minors of this new last row,

$$\sum_{r=1}^n (-1)^{r-1} f_r(x) W_r(x) = 0,$$

* Cf. *Orelle*, vol. 77 (1874), p. 248; also footnote, p. 284, of the article of the author, cited on p. 463.

or, since $W_n(x) \neq 0$ in $[a, b]$,

$$f_n(x) = \sum_{r=1}^{n-1} (-1)^{r+n-1} f_r(x) \frac{W_r(x)}{W_n(x)}.$$

The substitution of (6) in this last relation gives

$$(7) \quad f_n(x) = \int_a^x \frac{W(\xi) \left[\sum_{r=1}^{n-1} (-1)^{r+n-1} f_r(x) W_{n,r}(\xi) \right]}{[W_n(\xi)]^2} d\xi + \frac{\sum_{r=1}^{n-1} (-1)^{r+n-1} f_r(x) W_r(z)}{W_n(z)}.$$

It may be noted that the expression in brackets in the numerator of the integrand in (7) is equal to the determinant

$$\Delta_n(x, \xi) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_{n-1}(x) \\ -f_1(\xi) & -f_2(\xi) & \cdots & -f_{n-1}(\xi) \\ \vdots & \vdots & \cdots & \vdots \\ -\frac{d^{n-2} f_1(\xi)}{d\xi^{n-2}} & -\frac{d^{n-2} f_2(\xi)}{d\xi^{n-2}} & \cdots & -\frac{d^{n-2} f_{n-1}(\xi)}{d\xi^{n-2}} \end{vmatrix}.$$

To obtain the desired result we have now only to replace the elements $f_n(x_1), f_n(x_2), \dots, f_n(x_n)$ of the last column of D by the corresponding expressions (7), multiply the elements of the r th column by $(-1)^{r+n-1} W_r(z)/W_n(z)$ for each value of r from 1 to $n-1$, and subtract from the last column. This gives us

$$(8) \quad D = \begin{vmatrix} f_1(x_1) & f_2(x_1) & \cdots & f_{n-1}(x_1) & \int_a^{x_1} \frac{W(\xi) \Delta_n(x_1, \xi)}{[W_n(\xi)]^2} d\xi \\ f_1(x_2) & f_2(x_2) & \cdots & f_{n-1}(x_2) & \int_a^{x_2} \frac{W(\xi) \Delta_n(x_2, \xi)}{[W_n(\xi)]^2} d\xi \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ f_1(x_n) & f_2(x_n) & \cdots & f_{n-1}(x_n) & \int_a^{x_n} \frac{W(\xi) \Delta_n(x_n, \xi)}{[W_n(\xi)]^2} d\xi \end{vmatrix}$$

where z is any point of $[a, b]$.

Formula (8) may be transformed in a number of ways which we shall not discuss here. It should be remarked that

if the integral is taken in the ordinary or Riemannian sense, we must presuppose the integrability of the expressions $W(x) \cdot W_{n,r}(x) / [W_n(x)]^2$, but this is not necessary if the Lebesgue integral is used, provided, at least, the above expression is finite in $[a, b]$. The condition $W_n(x) \neq 0$ in $[a, b]$ may also be removed in certain cases, as when $W_n(x)$ vanishes at only a finite number of points, $x_0, x_1, \dots, x_m$ in $[a, b]$ so that $\lim_{r \rightarrow \infty} W_r(x) / W_n(x)$ exists and is finite for $(r=0, 1, \dots, n-1)$. It seems unlikely, however, that formula (8) can be so extended as to give new criteria for linear dependence.

In conclusion we note that formulas (3), or (3'), and (8) taken together express G in terms of W in such a way as to show that $G=0$ when W vanishes, under the restrictions named. We thus have what may be regarded as the converse of (2). Similarly (8) and (2) express D in terms of G .

NORTHWESTERN UNIVERSITY,
March 3, 1911.

NOTE ON INTEGRATION OF SERIES BY LEBESGUE INTEGRALS.

BY MR. W. A. WILSON.

LEBESGUE has shown in his work on integration that if a limited function, $f(x) \geq 0$, is measurable for a measurable field A , it is "summable," or possesses a Lebesgue integral, and the value of this integral is the measure of the ordinate set Y , whose points are defined by the conditions: x in A , $0 \leq y \leq f(x)$. The converse is also true; that is, if Y is measurable, $f(x)$ is measurable and

$$\text{meas } Y = \int_A f(x) dA.$$

A proof of this may be found in Schoenflies, *Jahresbericht der Deutschen Mathematiker-Vereinigung*, Ergänzungsband II, part II, page 320.

It is the purpose of this note to show how by use of this idea the proof of Lebesgue's theorem on termwise integration of series can be greatly simplified and reduced to elementary theorems on point aggregates. The theorem in question is proved in Hobson's *Theory of Functions of a Real Variable*,

§§ 381, 382, 383. In the following proof the field A is taken as a one-dimensional set for clearness, but the proof holds for any number of dimensions.

THEOREM: Let $\{f_n(x)\}$ be a sequence of functions, integrable (in the sense of Lebesgue) and converging to $f(x)$ at all points of the measurable set A . Let the $f_n(x)$ be uniformly limited in A . Then $f(x)$ is integrable and

$$\int_A f(x) dA \equiv \lim_{n \rightarrow \infty} \int_A f_n(x) dA.$$

Proof. Suppose first that each $f_n(x) \geq 0$ in A . Let Y_n be the set of ordinates corresponding to $f_n(x)$. Then Y_n is measurable and

$$\text{meas } Y_n = \int_A f_n(x) dA.$$

Let $U_n = \text{union of } Y_n, Y_{n+1}, \dots$. Then U_n is measurable and

$$(1) \quad \text{meas } U_n \geq \text{meas } Y_{n+p} \quad (p \geq 0).$$

Now let $D = \text{divisor } \{U_n\}$. Then D is measurable and

$$(2) \quad \text{meas } D = \lim_{n \rightarrow \infty} \text{meas } U_n \geq \overline{\lim}_{n \rightarrow \infty} \text{meas } Y_n.$$

Evidently D consists of a set of ordinates. For let (x_1, y_1) be a point of D . Then (x_1, y_1) lies in every U_n and hence in an infinity of the Y_n . Hence the points (x_1, y) , $y \leq y_1$ lie in an infinity of the Y_n ; hence in all the U_n and thus in D .

If any ordinate of D is not complete, add its upper limiting point and call the resulting set E . It is obvious that $\text{meas } E = \text{meas } D$. We now show that E is the ordinate set of $f(x)$ over A . Suppose the point (x_1, y_1) is the upper end of the ordinate of E through x_1 . We must prove

$$(3) \quad y_1 = y_2 = f(x_1).$$

Take $y = y_2 - \epsilon$. Since $f_n(x_1) \rightarrow f(x_1)$, for n_0 sufficiently great, the point (x_1, y) lies in all Y_n , $n \geq n_0$; hence in all the U_n and in E .

$$(4) \quad \therefore y_1 \geq y = y_2 - \epsilon.$$

Now consider $y = y_2 + \epsilon$. Since $f_n(x_1) = f(x_1)$, the point (x_1, y) lies in no Y_n for $n \geq n_0$; hence not in all the U_n and thus not in E .

$$(5) \quad \therefore y_1 \leq y = y_2 + \epsilon.$$

Relations (4) and (5) give (3).

Hence E is the ordinate set of $f(x)$ and (2) gives

$$(6) \quad \int_A f(x) dA = \text{meas } E \geq \overline{\lim}_{n \rightarrow \infty} \text{meas } Y_n.$$

Now set $D_n = \text{divisor } (Y_n, Y_{n+1}, \dots)$ and $U = \text{union } (D_1, D_2, \dots)$. The sets D_n and U are measurable;

$$(7) \quad \therefore \text{meas } U = \lim_{n \rightarrow \infty} \text{meas } D_n \leq \underline{\lim}_{n \rightarrow \infty} \text{meas } Y_n.$$

As before, U is an ordinate set and addition of upper limiting points of ordinates gives V of the same measure. In a similar manner to the case of E , we can show that V is the ordinate set of $f(x)$ and get, with (7)

$$(8) \quad \int_A f(x) dA = \text{meas } V \leq \underline{\lim}_{n \rightarrow \infty} \text{meas } Y_n.$$

Relations (6) and (8) give

$$(9) \quad \int_A f(x) dA = \lim_{n \rightarrow \infty} \text{meas } Y_n = \lim_{n \rightarrow \infty} \int_A f_n(x) dA.$$

When the $f_n(x)$ are unrestricted in sign, we can set

$$\phi_n(x) = f_n(x) + C, \quad \text{where } C \geq \max_{n, x} |f_n(x)|,$$

and apply (9) to $\phi_n(x)$, giving

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_A f_n(x) dA &= \lim_{n \rightarrow \infty} \int_A \phi_n(x) dA - \int_A C \cdot dA \\ &= \int_A \phi(x) dA - \int_A C \cdot dA \\ &= \int_A [\phi(x) - C] dA = \int_A f(x) dA. \end{aligned}$$

EISENHART'S DIFFERENTIAL GEOMETRY.

A Treatise on the Differential Geometry of Curves and Surfaces.
By LUTHER PFAHLER EISENHART. Boston, Ginn and Company, 1909. xii + 474 pages.

THE plan of this book is similar to that of a number of mathematical treatises which have recently been published in the United States. It is intended to serve as an introduction to the subject of differential geometry for use in graduate courses; at the same time it contains accounts of recent researches on special topics by the author and by other writers which will be of interest to any reader and may well serve as an inspiration to further investigation on the part of the student.

The writer of this review has found the book an exceedingly useful one to have in the hands of students in a course in differential geometry. It contains numerous examples, some inserted in the text to illustrate particular discussions, and others of a more general character at the end of each chapter. Many of these examples are difficult for the elementary student to handle, but a suitable selection can readily be made. In the course referred to the book was used for reference, for the examples, and as a basis for reports by students. It was not found practical to follow closely the arrangement of material or the details of the discussions of particular subjects. These are matters, however, which depend much upon the individual taste of the instructor and the character of the students with whom he has to deal, and the fact that the book presented in many cases a different view-point from that of the lecturer seemed to enhance rather than to diminish its value.

In these days of the popularity and elegant methods of the theory of functions of a real variable, it is interesting to note that much of the theory of surfaces can be developed without the use of imaginaries, and to see also that the existence and uniqueness theorems for real differential equations can be applied with economy in many places. Professor Eisenhart has made use from the start of expansions in series. It is true of course in some cases, as for example with minimal curves and surfaces, that the theory would be incomplete or difficult to phrase without the in-

introduction of analytic functions. But it is certainly doubtful in many others if any simplification is effected by their use.

One of the most interesting features of the book is the introduction of movable axes, the so-called "moving trihedrals," for twisted curves as well as surfaces. The fundamental rotations and translations associated with these trihedrals are freely applied in the writings of Darboux and others, and their applications rival in importance those of the Gaussian fundamental quantities for a surface. The student will hardly find himself at home in the literature of to-day without an intelligent acquaintance with them. Professor Eisenhart has developed the properties of these quantities without the help of the notions from mechanics used by Darboux, and has applied them in so many situations that their importance and uses seem perfectly clear to the reader.

The book would be somewhat more convenient if the chapter and section numbers had been printed at the top of each page, as there are numerous references to sections and to numbered equations in the different chapters which often require considerable turning. The typography is otherwise excellent, and a good index adds much to the usefulness of the work.

The contents of the book may be roughly divided into four parts. The first six chapters contain the fundamental principles of the theory of twisted curves and surfaces, and these are followed in Chapters VII and VIII by applications to a variety of special cases, including quadrics, ruled surfaces, minimal surfaces, surfaces of constant curvature, and surfaces with plane or spherical lines of curvature. One of the topics most fully treated is the theory of the deformation of surfaces in Chapters IX-XI. Chapters XII on rectilinear congruences, XIII on congruences of circles orthogonal to a family of surfaces, and XIV on triply orthogonal systems of surfaces form the concluding group and with Chapter VIII furnish most interesting reading.

The theory of twisted curves in space is developed in Chapter I. The osculating plane is defined as the limiting position of a plane through a tangent and a neighboring point M of the curve as M approaches the point of tangency, and the osculating sphere is defined as a sphere with its center in a proper position on the polar axis. It seems, in some ways more desirable to define these as limiting positions of a plane or sphere through three or four neighboring points of the curve. The

same methods may then be applied to both of them as well as to the determination of the tangent line and osculating circle, and the method itself is important in other connections.

One is impressed here at the start with the importance of the existence and uniqueness theorems for a system of differential equations. In discussing the determination of a curve by means of its intrinsic equations, as in many other places in the book, the author has such a system to consider. It would certainly mean economy of thought for the reader, and perhaps even economy in printed pages, if these theorems were stated somewhere in a form suited to the applications which are made of them. With them in mind the intrinsic equation theory could be quite concisely stated.

The sections devoted to the moving trihedral and its applications are interesting ones. The notions involved seem difficult for the elementary student to handle, but they are exceedingly important. Professor Eisenhart has derived the fundamental equations

$$\frac{\delta\xi}{ds} = \frac{d\xi}{ds} + 1 - \frac{\eta}{\rho}, \quad \frac{\delta\eta}{ds} = \frac{d\eta}{ds} + \frac{\xi}{\rho} + \frac{\zeta}{\tau}, \quad \frac{\delta\zeta}{ds} = \frac{d\zeta}{ds} - \frac{\eta}{\tau}$$

by means of expansions in series. The quotients in the first members of these equations determine the direction of the absolute motion in space of the point whose coordinates are ξ, η, ζ with respect to the moving axes, while those in the second members give the motion relative to the moving axes themselves. If the coordinates ξ', η', ζ' with respect to a fixed trihedral at a point M' of the curve are expressed in terms of the coordinates ξ, η, ζ with respect to a trihedral with vertex at a neighboring point M , the formulas just given can be very simply derived by differentiation and use of the initial values given by Professor Eisenhart on page 30. The use of symbols for differentiation in the left members of the equations above is unfortunate. The direction $\delta\xi/ds : \delta\eta/ds : \delta\zeta/ds$ cannot be found by differentiation of ξ, η, ζ and it would be better to denote it by other symbols, say $V_\xi : V_\eta : V_\zeta$. A number of applications of the formulas above have been made in the text. The properties of involutes and evolutes of a curve can also be derived in this way, as is suggested in example 8, page 47, apparently more simply than by the methods used in § 21.

This chapter contains also a short discussion of the theory of the integration of a Riccati equation, a study of developable

surfaces as the tangent surfaces of curves, and a section on minimal curves.

The more elementary portions of the surface theory are presented in Chapters II–IV. In Chapter II the notion of curvilinear coordinates is introduced, and the theory of envelopes with applications to developable surfaces and envelopes of families of spheres is explained. The discussion of the determination of characteristic curves in terms of the moving trihedral would be clearer to the reviewer if the method suggested above for the derivation of the fundamental equations in Chapter I had been used.

In Chapter III the quantities E, F, G are defined and those parts of the surface theory which depend upon them exclusively are developed. It is to be regretted that a precise definition of a differential parameter was not set down. Roughly expressed, a differential parameter is a function of $E, F, G, \phi, \psi, \dots$ and their derivatives with respect to u, v , which remains unchanged in form when these functions are replaced by their equivalents in terms of E_1, F_1, G_1 and derivatives with respect to two new variables u_1, v_1 . According to this definition the functions E, F, G are not themselves invariants, as is stated at the bottom of page 87. And it also does not seem to be true that a function expressible in terms of differential parameters with respect to one system of coordinates will be equal to the same expression formed with respect to another, unless the function is itself invariant. The proof given on page 165 for the invariance of $\Delta_{xx}\theta$ would therefore seem to be only a proof of formula (38) for expressing $\Delta_{xx}\theta$, known by other means to be invariant, in terms of other differential parameters.

In Chapter IV the second fundamental quantities D, D', D'' appear for the first time, and the well known relations between the curvatures of lines through a point on a surface are set down. The reader is also introduced to lines of curvature and conjugate systems, asymptotic lines, geodesics, and the spherical representation of a surface. These conceptions appear again in Chapter VI for a more detailed discussion with the help of the Christoffel symbols, the moving trihedral, and the fundamental equations derived in Chapter V. On page 117 the formula near the bottom of the page should read

$$\frac{d^2x}{ds^2} = \frac{\partial^2x}{\partial u^2} \left(\frac{du}{ds} \right)^2 + 2 \frac{\partial^2x}{\partial u \partial v} \frac{du}{ds} \frac{dv}{ds} + \frac{\partial^2x}{\partial v^2} \left(\frac{dv}{ds} \right)^2 + \frac{\partial x}{\partial u} \frac{d^2u}{ds^2} + \frac{\partial x}{\partial v} \frac{d^2v}{ds^2},$$

though the argument which follows is unaltered by the omission of the last two terms. On page 123 the statement that a surface lies on only one side of the tangent plane at points of parabolic curvature is evidently incorrect. This is the old question of the minimum of a function whose expansion begins with a quadratic form having a vanishing discriminant. One needs only to consider the surface generated by revolving a sine curve about a tangent at one of its maximum points. The lines along which the curvature is parabolic are generated by the points of inflexion.

In the next chapter the equation of Gauss expressing the curvature of a surface in terms of ε , $\mathcal{F}$, $\mathcal{G}$ and their derivatives is derived, as well as the two so-called Codazzi equations, and a proof is given of the fundamental theorem of surface theory. This is the theorem which states that any six functions E, F, G, D, D', D'' which satisfy the Gauss and Codazzi equations are the fundamental quantities of a surface which is uniquely defined except for its position in space. Other forms of the theorem are also given in which the surface is characterized by D, D', D'' and the functions $\varepsilon, \mathcal{F}, \mathcal{G}$ defining its spherical representation, or by the direction cosines X, Y, Z of its normal together with the distance W of the tangent plane from the origin. The functions X, Y, Z, W are called the tangential coordinates of the surface.

A surface may also be characterized by functions of u, v representing the rotations and translations of a system of axes which move so that the xy -plane is always tangent to the surface at the origin, and in such a way that the x -axis always makes an angle $U(u, v)$ with the curves $v = \text{constant}$. This system of axes is called the moving trihedral of the surface. Professor Eisenhart has derived the equations of condition which must be satisfied by the six rotations p, q, r, p_1, q_1, r_1 and the four translations ξ, η, ξ_1, η_1 of the moving trihedral with the help of the moving trihedral for a curve already discussed in Chapter I, and his method seems simpler than that of Darboux. Any ten functions $p, q, r, \dots, \eta$ which satisfy these equations define a moving trihedral enveloping a surface which is uniquely determined except for its position in space. The latter part of the chapter is devoted to the determination of the principal properties of a surface in terms of the rotations and translations above mentioned, and to the application of these results in the cases of parallel surfaces and evolutes of a given surface.

With the formulas of the preceding chapter at hand it is possible for the author, in Chapter VI, to study more comprehensively the theory of systems of curves on a surface. In the first half of the chapter the conditions are derived which must be satisfied by functions E, F, G in order that the parametric lines may be asymptotic, or by $\mathcal{E}, \mathcal{F}, \mathcal{G}$ in order that the parametric lines on the sphere may represent a system of asymptotic or isothermal conjugate lines on the surface, and in each case it is shown how the surfaces themselves may be determined. A number of theorems are also given showing how surfaces with asymptotic or conjugate parametric lines are determined by the solutions of certain partial differential equations of the second order. The last half of the chapter is devoted to geodesics and geodesic coordinate systems. A proof is given of the celebrated theorem of Gauss concerning the sum of the angles of a geodesic triangle, and the minimizing properties of geodesics are briefly discussed from the standpoint of the calculus of variations.

The next two chapters contain applications of the theory of surfaces to special cases. The properties of systems of lines of various kinds on quadrics and ruled surfaces is developed, and an interesting discussion of the more important properties of minimal surfaces is given. In Chapter VIII surfaces of constant curvature are treated in the first part. There are four kinds of transformations of such surfaces, called the transformations of Hazzidakis, Bianchi, Bäcklund, and Lie. The result of these transformations is the determination of surfaces of constant curvature having the same curvature as that of the given one and related to it by some geometrical or analytical process. Thus for the transformation of Bianchi it is found that if circles of radius a are described in the tangent planes to a surface of constant curvature $-1/a^2$, with centers at the points of contact, then these circles are the orthogonal trajectories of an infinity of surfaces of curvature $-1/a^2$. The transformation of Bäcklund is a generalization of this transformation, while those of Hazzidakis and Lie are analytical in character. It may be of assistance to the reader to remark that §§ 118–119 are much clearer if reference is made to § 77, page 190, and § 82, pages 199, 200. In fact the argument does not seem to be complete without these references, though no mention is made of them. W -surfaces are surfaces for which a relation of some sort exists between the principal radii of curvature. They include mini-

mal surfaces and surfaces of constant mean curvature. After discussing them the author devotes the last part of Chapter VIII to surfaces which have systems of lines of curvature of special kinds. Examples of these are the surfaces of Monge which have one system of plane geodesic lines of curvature, and which are generated by a plane curve whose plane rolls without slipping over a developable surface. The equations at the bottom of page 313 seem to be misprinted. They should read

$$\xi = \frac{v}{2(1-u^2)^{\frac{1}{2}}} [u^2 + \frac{1}{2} + a - 2(1+u^2)^{\frac{1}{2}}],$$

$$\zeta = \frac{1}{2(1-u^2)^{\frac{1}{2}}} [u^2(u^2 + \frac{1}{2} + a) - (u^2 - \frac{1}{2})(1+u^2)^{\frac{1}{2}}],$$

$$\rho = \frac{1}{2}(u^2 + \frac{1}{2} + a)(1-u^2)^{\frac{1}{2}}.$$

The chapters in which the deformation of surfaces is discussed have been mentioned above. In determining all the surfaces with a given linear element

$$Edu^2 + 2Fdu\,dv + Gdv^2$$

by the method of Darboux, it is found that whenever a solution $z(u, v)$ of the equation

$$\Delta_{22}z = (1 - \Delta_1 z)K$$

is known, where the differential parameters and curvature K are formed with respect to the quadratic form just given, then two other functions $x(u, v)$, $y(u, v)$ can be determined by quadratures, which with $z(u, v)$ define a surface having the length element required. The method of Weingarten, published in a memoir which was awarded the grand prize of the French Academy in 1894, makes use of the moving trihedral. A set of coordinates on the surface can be so chosen that

$$r = v, \quad r_1 = 0, \quad K\sqrt{EG - F^2} = 1,$$

where r and r_1 are two of the rotations of the trihedral. The equations which must be satisfied by the translations of the trihedral are then

$$\frac{\partial \xi}{\partial v} - \frac{\partial \xi_1}{\partial u} = -v\eta_1, \quad \frac{\partial \eta}{\partial v} - \frac{\partial \eta_1}{\partial u} = v\xi_1,$$

$$p\eta_1 - \eta p_1 + \xi q_1 - q\xi_1 = 0.$$

The last equation can be expressed entirely in terms of u and differential parameters of u with respect to the length element of the spherical representation of the x -axis of the moving trihedral. When this is done, any other solution $u_1(u, v)$ of the differential equation so found furnishes a surface applicable to the original one. In the latter part of the chapter the results of Weingarten are given a different form and applied to the determination of surfaces applicable to surfaces of revolution and other special cases. It is not clear from the text that the functions ξ, η, ξ_1, η_1 can be chosen in the form (29) of page 357 for any surface, as is implied in the theorems of page 360.

If a family of surfaces of the form

$$x' = x(u, v) + \epsilon x_1(u, v) + \epsilon^2 x_2(u, v) + \dots,$$

$$y' = y(u, v) + \epsilon y_1(u, v) + \epsilon^2 y_2(u, v) + \dots,$$

$$z' = z(u, v) + \epsilon z_1(u, v) + \epsilon^2 z_2(u, v) + \dots,$$

were such that each had the same length element, then any one of them would be applicable to every other. The problem of the so-called infinitesimal deformation of the surface $S(\epsilon = 0)$, is the determination of the first coefficients x_1, y_1, z_1 in such an expansion. It turns out that the problem is equivalent to the determination of a surface S_1 corresponding to S in such a way that corresponding curves on the two are orthogonal, and this depends upon the solution of a partial differential equation of the second order. The latter problem is again equivalent to the determination of a surface S_0 associate to S , i. e., one such that tangent planes at corresponding points of S and S_0 are parallel, and such that asymptotic lines on either surface correspond to a conjugate system on the other. These results are derived in Chapter XI with further interrelations between the triples of surfaces S, S_0, S_1 .

One of the most interesting applications of differential geometry, on account both of the elegance of the theorems and also of the opportunities for the exercise of geometric imagination which present themselves, is the theory of congruences, to which Chapter XII is devoted. Through any one of a double infinity of straight lines there pass an infinity of ruled surfaces belonging to the congruence, whose lines of striction all cut the given line on a certain segment. The end points of the segment correspond in this way to two principal ruled surfaces whose tangent planes at the end-points are perpendicular to

each other. Similarly through each line there pass two developable surfaces whose cuspidal edges are tangent to the line at two focal points equidistant from the end-points just mentioned. The totality of the focal points on all the lines of the congruence form two focal surfaces. After developing these conceptions and their interrelations with each other, the author takes up the study of special congruences characterized for the most part by restrictions of various kinds placed upon their focal surfaces. Thus pseudospherical congruences have pseudospherical focal surfaces, and congruences of Guichard are those whose developables touch the focal surfaces along lines of curvature.

A cyclic system is a congruence of circles which has a single infinity of orthogonal surfaces. With every such system is associated a congruence of lines, the axes of the circles, called a cyclic congruence. Conversely, with every cyclic congruence there is associated but one cyclic system, unless the congruence is of the special type called congruences of Ribaucour. In § 178 Professor Eisenhart has exhibited a transformation of surfaces which have the same spherical representation of lines of curvature as pseudospherical surfaces, into others of the same type. This is a generalization due to the author of the transformation of Bianchi for pseudospherical surfaces.

The concluding chapter of the book is devoted to triply orthogonal systems of surfaces, examples of which are provided by the orthogonal surfaces of a cyclic system, taken with the two families of surfaces generated by the circles of the system which intersect them in their lines of curvature. The determination of triply orthogonal systems which have one family consisting of surfaces of revolution or of pseudospherical surfaces, is considered, and it is shown, following Darboux, that the general problem of the determination of all triply orthogonal systems may be reduced to the solution of a partial differential equation of the third order.

GILBERT AMES BLISS.

January 4, 1911.

STUYVAERT'S STUDIES IN ANALYTIC
GEOMETRY.

Cinq Études de Géométrie analytique. Par M. STUYVAERT.
Gand, E. Van Goethem, 1908. vi + 230 pp.

THIS book, consisting of applications of the theory of matrices and elimination, was awarded the "Prix François Deruyts" by the Royal Academy of Belgium and forms an important addition to the methods of analysis used in geometry. Whether it is a question of elimination or superelimination (the study of the conditions that two equations have more than one root in common), the result leads to the vanishing of a rectangular matrix, i. e., to the existence of a certain linear relation or "faisceau" of such between the elements of each row or column, and from the details of structure of this matrix can be read the properties of the variety represented by equating it to 0. The results are in general known, so the author seldom trusts the reader to discover the new theorems, but instead labels them as such. The originality and value of the book consist in the methods used, and it is surprising with what simplicity and elegance the machinery works.

The first of the five studies of which the book consists ("Applications géométriques de la théorie des matrices") investigates those processes of elimination which give rise to matrices and the use which can be made of this theory for the determination of special elements of geometrical figures. The conditions that two equations in x have at least two common roots can be expressed by the vanishing of a matrix in l lines and $l + 1$ columns. If the elements of such a table are ternary or quaternary forms, we have respectively the representation of a finite number of points or a skew curve. Such a representation of a curve gives at once, by means of tables obtained from the original matrix by the suppression or adjunction of various lines and columns, its order, genus, modes of generation, circumscribing surfaces, multiseccant curves, groups of remarkable points, and other properties.

After deriving general formulas for the orders and genus of curves represented by vanishing matrices the author discusses skew curves of orders three to ten. The following will illu-

strate his methods: If $a_x, \dots$ represent linear quaternary forms, the relations

$$\begin{vmatrix} a_x & b_x & c_x \\ a'_x & b'_x & c'_x \end{vmatrix} = 0$$

represent the most general skew cubic. Its system of bisecants is $\lambda a_x + \mu b_x + \nu c_x = 0$, $\lambda a'_x + \mu b'_x + \nu c'_x = 0$. If we precede the above matrix by a line of constants α, β, γ , the resulting equation gives ∞^2 quadrics circumscribing the curve. Similarly

$$\begin{vmatrix} a_x & b_x & c_x \\ a'_x & b'_x & c'_x \\ a''_x & b''_x & c''_x \end{vmatrix} = 0$$

is a skew C_6 of genus 3. As the four equations of the type $\lambda a_x + \lambda' a'_x + \lambda'' a''_x = 0$ are compatible in λ for all points of C_6 , the latter is the locus of intersections of homologous planes of four collinear "gerbes." In a similar way we can see that it is the locus of points of intersection of homologous rays of three superimposed projective spaces. If we precede the matrix of C_6 by a column of constants λ , the resulting table annuls itself for four points of C_6 . These four points annul an evident determinant containing eight arbitrary constants whose vanishing gives a quadric, which is seen to be circumscribed to a variable skew cubic cutting C_6 eight times, and whose equation comes from the above determinant. Thus if we vary $\lambda, \lambda', \lambda''$, we have on C_6 , ∞^3 quadruples of points such that each of these groups can be joined by a quadric to all skew cubics cutting C_6 eight times.

When matrices of l lines and $l + 1$ columns of independent linear forms are used, the resulting curve is of order $\frac{1}{2}l(l + 1)$; hence in order to discuss skew curves of order 5, 7, 8 and 9 the author introduces non-linear forms, e. g., C_6 is

$$\begin{vmatrix} a_x^2 & b_x & c_x \\ a_x'^2 & b'_x & c'_x \end{vmatrix} = 0.$$

The fact that non-linear forms had to be introduced suggests one of the most interesting problems for research in this subject namely: "The determination of what curves are representable by matrices."

The section on the Jacobian of a system of three or five surfaces, i. e., the skew curve defined by the vanishing of the usual Jacobian matrix is of value from the standpoint of results, as the author's theorems are more complete than those of Cremona or Salmon. The problem of geometrical loci, i. e., the problem of eliminating t between $F(x, y, z, t) = 0$ and $f(x, y, z, t) = 0$, where the common roots in t may have various multiplicities, is well treated and general results are derived. The application of these results as a corollary to the very important subject of envelopes is however disappointing, being confined to an example. The study closes with sections on the multiseccants of rational curves, the multiseccant conics of skew curves, the elimination of two or more unknowns from a system of equations, and applications to space of four dimensions. The latter suggests another line for investigation, for when the number of homogeneous variables surpasses four, the algebraic problems are greatly changed, so that for example it is necessary to study matrices the difference of the numbers of whose columns and lines is greater than two, and also matrices whose elements are dependent forms.

In the first study the author limits himself to matrices whose elements are forms in a single variable. The case where they are forms of two or more series of variables and its application to particular cases furnishes the basis of a theory of congruences of skew curves, a theory which is also still in the making and is the subject of Study II: "Congruences de variétés algébriques annulant des matrices." The author considers the matrix of l lines and $l + 1$ columns containing homogeneous forms in $x_1, \dots, x_n$ and $\alpha_1, \alpha_2, \alpha_3$. Its vanishing represents a congruence of varieties of $d - 3$ dimensions in a space of $d - 1$ dimensions, and the writer seeks under the easiest hypotheses how α must enter in order that an arbitrary system of values of x belong to a single variety of the congruence. After general theorems he limits himself to matrices of two lines and three columns of forms linear in both x and α , and classifies the linear congruences so obtained into six types. If $d = 4$ we have a theory of linear congruences of skew cubics and if $d = 3$ of linear congruences of coplanar triangles.

The author defines a singular point as one belonging to a simple infinity of members of a congruence and discusses the singular points in the above types and the figures they represent, cleverly reducing this problem to a study of the loci repre-

sented by the formulas distinguishing the different types of congruences; for example, if $d = 4$, we have for type III, defined by

$$\left\| \begin{array}{ccc} \alpha_1 a_x + \alpha_2 b_x & \alpha_1 a'_x + \alpha_2 b'_x & \alpha_1 a''_x + \alpha_2 b''_x \\ \alpha_1 d_x + \alpha_2 g_x & \alpha_1 d'_x + \alpha_2 g'_x & \alpha_1 d''_x + \alpha_2 g''_x \end{array} \right\| = 0,$$

a congruence of skew cubics having for directrices a sextic of genus 3 and two skew cubics, each curve of the congruence and the two cubic directrices cutting the sextic eight times and each curve cutting the two cubic directrices once.

The third study, "La théorie des matrices dans l'espace réglé" is very short but suggestive of what more can be done. It contains sections on matrices with l lines and $l + 3$ columns (whose vanishing as soon as the number of variables surpasses four is in general compatible), congruences of straight lines annulling matrices, vanishing of the first minors of a determinant, and complexes annulling a determinant. The matrix M of l lines and $l + 1$ columns whose elements are homogeneous forms of plückerian line coordinates when set equal to zero represents a congruence Γ of straight lines. If we precede M by a line of constants α , we get a complex C_α including Γ . Two such complexes C_α and C_β have Γ and a congruence Δ in common, the latter annulling the matrix obtained by preceding M with the lines α and β . Δ and Γ have a ruled surface in common.

The finding of the points which annul all the first minors of a determinant (x_k) ($i, k = 1, 2, 3$) is reduced to finding the points common to two vanishing matrices which represent surfaces. If we have variables $x_1, \dots, x_d$ the vanishing of all the first minors of a determinant of nine elements gives in space of $d - 1$ dimensions a variety of $d - 5$ dimensions. If in the above determinant the variables are again line coordinates, its vanishing gives a complex C whose double elements annul all the determinant's first minors and belong to ∞^1 ruled surfaces. If the determinant is preceded by a line of constants we have a congruence and so again get an interesting relation between complexes, congruences, and ruled surfaces.

Study IV, "Sur une forme doublement quadratique binaire et symétrique" serves as an introduction to the last article and concerns a form which in non-homogeneous variables is

$$\phi = a - x^2 y^2 + bxy + c + h(xy)(x + y) + g(x^2 + y^2) + f(x + y).$$

In this form x and y stand for abscissas of two "ponctuelles" and not cartesian coordinates. ϕ is used for a classification of symbolic forms and finally applied to conics and surfaces, where the author deals chiefly with groups of elements harmonically situated, such treatment having been suggested by Halphen who in his *Traité des fonctions elliptiques* reduced the study of two coplanar conics to that of a form similar to ϕ , whose geometric meaning was that it set up the most general correspondence on a conic by means of tangents to another conic. The study closes with ruled surfaces having for generators the bisecants of a skew cubic and as a special case an interesting surface of the fourth order defined, for example, as the envelope of quadrics determined by couples of homologous elements of two projective series of bisecants.

If ϕ ceases to be symmetric and x, y are coordinates in a plane, the author has a basis for his last study "*Quadrilatères de Steiner dans certaines courbes et surfaces algébriques*" which is remarkable in the beauty and symmetry of its analysis and theorems. The author first proves the celebrated theorem of Steiner, "A plane binodal quartic is not in general circumscribed to a quadrangle having two of its diagonal points at the nodes; but if there exists one such quadrangle there is an infinity, in which case the third diagonal point describes a conic and the sides of the quadrangle which do not pass through the nodes envelope a curve of the fourth class," and calls such quadrangles "quadrillée." The necessary and sufficient conditions that a quartic be quadrillée are derived as well as many theorems on quadrillée quartics of two, three and four nodes.

The author next seeks, by original methods, the properties to the skew quartic of the first kind C'_4 which correspond of those explained for plane quartics, using in his analysis quadrillée quadrics (quadrics of Voss) and quadrillée cones, i. e., perspective cones of C'_4 every plane trace of which is a quadrillée quartic. We shall note one theorem as it is proven without the customary use of elliptic functions. "Through every skew C'_4 there pass six quadrillée quadrics. In the system $F'_4 + kS'_4$ having C'_4 as a base there are four cones defined by a biquadratic form in k and the six quadrillée quadrics annul the sextic covariant of this form." This section is followed by one on special surfaces of the fourth order having a simple infinity of double points. These double points can be divided in general into ∞^2 couples and through each of these couples pass six

quadrillée sections whose ∞^2 planes envelope a surface. The ∞^3 diagonals of the quadrangles generate a complex and the third diagonal points are on ∞^2 conics. The book closes with some paragraphs on graphical tables for the representation of functions, where the author solves the problem which when proposed to him suggested this last study, namely, to determine when the equation of a certain surface of the sixth order could be put in the form of a determinant equated to zero.

The very power of Professor Stuyvaert's methods, which cover every page with italicized theorems, makes the book monotonous to read, but at the same time gives it value as a compendium. As we have pointed out throughout this review, the author of these five studies has made a large addition to the instruments of a modern geometer and at the same time pointed out the way to other fields of investigation, perhaps the best of which is to discover how to do away with the limitations of the above methods in which to a large extent the curve has to be found to fit the formulas.

E. GORDON BILL.

SHORTER NOTICES.

Études sur Léonard de Vinci, ceux qu'il a lus et ceux qui l'ont lu. Par PIERRE DUHEM. Paris, A. Hermann, 1906, 1909. 8vo. 2 volumes. x + 355, iv + 469 pp. 27 francs.

It would be difficult to find a man better fitted for the preparation of a work of this character, either by temperament or because of scholastic attainments, than the eminent professor of theoretical physics of the Faculty of Sciences at Bordeaux. Writing of a great genius whose wide range of information covered all that was then known of physics, of a man whose interest in France showed itself in divers activities, of a scholar who was in correspondence with the learned men of his time, and of an inventor whose fertile imagination was seen in a large number of helpful devices, M. Duhem had a most inspiring subject and one which he could hardly have failed to treat *con amore*.

The author confesses at the beginning to the feeling of awe that such a topic would naturally create in any serious mind that contemplates the work of this master. In his own case, he tells us, this was followed by a period of contemplation, more free from emotion, in which the idea was prominent that

such a genius was after all a human being, that he was born as other men and that he struggled as others struggle, and that the problem is rather to mark the footsteps by which he advanced than to stand in awe of his great attainments. Well does he exclaim, however, "Mais ce récit, combien il est difficile de l'obtenir exact et précis."

Of the labor expended in examining the materials for this work everyone familiar with Leonardo da Vinci is aware. He left a large number of manuscripts, and in addition to those that are fairly complete there are many brief memoranda, written in his usual manner from right to left, difficult to decipher, and so mixed up as to make their sequence hopelessly obscure. It is out of this mass of material that M. Duhem has endeavored to extract the essence, and to interpret it in the light of the works of the predecessors of Leonardo, and the work of his followers in the light of these notes. From the nature of the case, therefore, this work cannot be looked upon as a life of Leonardo, nor even as a dissertation upon his works and his discoveries. It is merely what its title asserts, a set of studies,—an inquiry into the works which he had read and which had helped him attain his high position, and into the works of those who had in turn been assisted by his own fertile mind in perfecting their labors.

The author begins by a chapter setting forth the influence of Albert of Saxony upon Leonardo da Vinci, particularly as shown in the former's *De cælo et Mundo*. He shows by one of Leonardo's manuscripts,* begun in September, 1508,† that this work was familiar to the young scientist, and that the latter had, indeed, a copy then in hand. This point is evidenced by his note, "Alberto decelo e mundo — da fra bernardino," the copy either belonging to or having been made by some Brother Bernardinus. The interesting thing, however, is the manifest influence of Albertus upon Leonardo, as is shown in numerous quotations from each with respect to the spots on the moon, center of gravity, and the sphericity of the earth. M. Duhem shows that Leonardo was indebted to his predecessor for numerous initiatives, but that in nearly every case he developed an independent theory of his own. Thus, although he was led by Albertus to study the center of grav-

* Lettered F in the Bibliothèque de l'Institut at Paris, and published by Ch. Ravaisson-Mollien in 1889.

† "Comincato ammilano addi 12 dissettembre 1508," as the manuscript states.

ity of the pyramid, the theory itself was developed by him, and in it he anticipated by half a century the labors of Maurolycus and Comandinus. Furthermore, as early as 1508, when Copernicus was only beginning his study of the solar system, Leonardo was already setting forth his views in opposition to the geocentric theory of the universe. In the same year, two generations before Palissy preached the doctrine, Leonardo formulated the principles that were afterwards accepted by the world as to the origin of fossils. In view of the testimony that M. Duhem adduces from the original manuscripts, M. Ravaisson's remark is not an exaggeration, that Leonardo should be looked upon as "le grand initiateur de la pensée moderne."

The second chapter treats of the indebtedness of the Spanish Jesuit Villalpanc to the genius of Leonardo. Villalpanc was born at Cordova in 1552 and died at Rome in 1608. The fact is established that Leonardo anticipated by a century certain discoveries of Galileo, that his knowledge of mechanics was remarkable for the time, and that he had apparently mastered all the works of his predecessors upon this subject. It is then shown that Villalpanc's leading theorems upon the subject were apparently taken from or inspired by the manuscripts of Leonardo.

The third chapter is devoted to the work of Bernardino Baldi (1553-1617), one of the most versatile scholars of his age. M. Duhem shows that he drew very largely upon the writings of his predecessor, and particularly in his theories of vortices, of the center of gravity, and of mechanics in general.

Of still greater interest is the fourth chapter, in which the indebtedness of Roberval and Descartes is shown. This does not relate to pure mathematics, however, but only to questions of mechanics.

The fifth chapter relates to the works of Thimon (Temo Judæus, of the fourteenth century) and their influence upon Leonardo's theory of tides, and upon his general view concerning the laws of hydrostatics and hydrodynamics.

This is followed by a chapter upon the influence of Leonardo on Cardan (especially in *De Subtilitate Libri XXI*) and Bernard Palissy (in the theory of fossils). M. Duhem's conclusion as to Cardan is what we might expect: "Comme Villalpanc, comme Bernard no Baldi, comme tant d'autres de ses contemporains, Cardan fut un plagiaire; mais en plagiant les idées de

Léonard de Vinci, il les sauva de l'oubli ; grâce à la grande vogue de son livre étrange, il les sema partout, et son manque de scrupules leur fit produire les découvertes dont elles portaient le germe. Celui qui mène les pensées humaines fait servir au progrès de la Science les plus tristes faiblesses des savants."

The first volume closes with a chapter on the contributions of Leonardo to the theory of gravity, and one containing an excellent biography of Albert of Saxony.

The second volume, which appeared in 1909, three years after the first, contains four parts. The first of these relates to "les deux infinis," and discusses the theory of the infinite and the infinitesimal according to the ideas of Aristotle and the schoolmen, closing with extracts from Leonardo's notes upon the subject. The reader naturally looks to see if the latter give any evidence of anticipating Cavalieri, but if he looks with expectation he is disappointed, for Leonardo explicitly takes the opposite view of geometric magnitudes. No idea of an integral calculus shows itself; and since Leonardo's entire treatment relates to physics rather than to mathematics, one can hardly be surprised.

The second chapter relates to the question of the plurality of worlds. In this Leonardo combats the assertion of Albertus Magnus that such plurality is impossible. To such a position he was undoubtedly led by the works of Nicolaus von Cusa (Nicolaus Cusanus, Nicholas of Cusa, 1401-1464), to whose contributions the third chapter is devoted. This chapter contains one of the best biographies of this great savant of the fifteenth century to be found anywhere. It is particularly valuable as showing both the influences that made Nicolaus the scholar that he was, and those that his writings exerted upon Leonardo.

The work closes with the contributions of Leonardo to geology, already mentioned incidentally in the first volume.

To the study of pure mathematics Leonardo da Vinci cannot be said to have contributed in any serious way. We have, it is true, some interesting geometric proofs that are due to him, and it is well known that geometry appealed to him, but his bent of mind was rather towards mechanics and the underlying philosophy of applied mathematics than towards any form of analysis. It is in the theory of attraction, in hydrodynamics, in the beginnings of celestial mechanics, in the theory of perspective, and in related topics, that the contributions of Leonardo are to be found.

The work of M. Duhem is a monumental one and it is deserving of great commendation. He has made the learned world his debtor by this labor of love. He has not written a history of science, but he has composed a work of the kind that makes the history of science possible.

DAVID EUGENE SMITH.

The Fundamental Theorems of the Differential Calculus. By W. H. YOUNG. Cambridge University Press. ix + 72 pp.

As stated in the preface, "rigidity of proof and novelty of treatment have been aimed at rather than simplicity of presentation, though this has never been lightly sacrificed." The chapter headings are: I. Preliminary notions. II. Limits. III. Continuity and semicontinuity. IV. Differentiation. V. Indeterminate forms. VI. Maxima and minima. VII. The theorem of the mean. VIII. Partial differentiation and differentials. IX. Maxima and minima for more than one variable. X. Extensions of the theorem of the mean. XI. Implicit functions. XII. On the reversibility of the order of partial differentiation. XIII. Power series. XIV. Taylor's theorem.

The ϵ argumentation usually found in such books is entirely absent. The notion of a limit point of a set of points is taken for granted and by means of it the whole theory of limits is constructed. Infinity is included among points approached as a limit point and hence without further particular statement functions are permitted to approach infinity as a limit the same as any other value. Following Baire, functions are considered as approaching multiple limits instead of one unique limit. Hence it comes about that many theorems which we are wont to see stated for certain classes of functions in terms of the equality of unique limits are here stated for more general classes of functions in terms of the equality or inequality of the upper or lower limits approached. As an example we select the following:

If, as x approaches the values a , $f(x)$ and $F(x)$ have both the unique limit zero, or $+\infty$, or $-\infty$, then the limits of $f(x)/F(x)$ lie between* the upper and lower limits of $f'(x)/F'(x)$, provided

A. a is not a limiting point of common infinities of $f'(x)$ and $F'(x)$;

* x is said to lie between a and b if $a \leq x \leq b$.

B. α is not a limiting point of zeros of $F'(x)$ unless these zeros are also zeros of $f'(x)$ and

C. $F(x)$ is monotone.

There are several departures from the usual treatments. Thus Baire's definition of upper and lower semicontinuity is changed in conformity with the definitions used by Young in the *Quarterly Journal*, volume 39, pages 67-83. The indeterminate forms $0/0$ and ∞/∞ are treated without the use of the mean value theorem, as is also Taylor's formula with a remainder.

In the chapter on Taylor's series the necessary and sufficient condition for the expansibility of a function of one real variable is stated to be that $y^{n+p} f_{(a+x)}^{(n)}/(n+p)!$ shall be bounded for all values of n, p, x (under certain restrictions). This seems to be a simplification of the well known condition of Pringsheim that the expression given must approach zero uniformly for certain y and x . It is curious that neither Pringsheim nor Young has hit upon the obvious condition, viz., that for some value of h , $h^n f_{(a+x)}^{(n)}/n!$ must be bounded or approach zero uniformly, as the case may be.

There is throughout an endeavor to make the treatment very general. This often makes the theorems clumsy and the proofs complicated. Necessary and sufficient conditions seem to be insisted on wherever such have been found.

The treatment not being designed for a first reading of the subject, many terms are left undefined, sometimes even to the confusion of the initiated. It is not surprising that semicontinuity (though upper and lower semicontinuity are defined) and uniform convergence of series are not defined, but it might well have been stated that a stretch means a rectangle.

Misprints have been noted as follows: page 7, 2d line, for *same* read *every*. Page 12, last line, insert exponent n . Page 32 in the statement of the theorem for $f^{(n)}(a + \theta h)/n!$, read $h^n f^{(n)}(a + \theta h)/n!$. Page 37, the "equation to zero" had better read the "equating, etc."

In several instances an interval is said to be closed when there is no need for such restriction (cf. the definition of continuity on page 8).

The reviewer believes greater simplicity as well as generality could be achieved in some instances by considering multiple valued functions as well as multiple valued limits. A possible further simplification of the condition for the expansibility of a function in Taylor's series has already been noted.

There are two appendices, one containing explanatory notes regarding certain theorems on sets of points which are involved in the treatment in the text and the other a short bibliography of recent papers by Young, Baire, Lebesgue, and others. Only twenty-five titles are given.

N. J. LENNES.

NOTES.

THE Eighteenth Summer Meeting of the AMERICAN MATHEMATICAL SOCIETY will be held at Vassar College on Tuesday and Wednesday, September 12-13. Titles and abstracts of papers intended for presentation at this meeting should be in the hands of the Secretary by Saturday, August 26.

THE April number (volume 12, number 3) of the *Annals of Mathematics* contains the following papers: "On the solutions of ordinary linear homogeneous differential equations of the third order," by G. D. BIRKHOFF; "Approximate representation," by W. E. BYERLY; "Note on cubic equations and congruences," by L. E. DICKSON.

At the meeting of the London mathematical society held on April 27, the following papers were read: By G. T. BENNETT, "On the geometry of a deformable octahedron"; by W. P. MILNE, "A symmetrical method of apolarly generating cubic curves"; by G. N. WATSON, "The solution of the homogeneous linear difference equation of the second order (second paper)"; by G. B. MATHEWS, "A cartesian theory of complex geometrical elements of space"; by A. CUNNINGHAM, "The number of primes of given linear form"; by M. J. M. HILL, "On the proofs of the properties of Riemann's surfaces discovered by Lüroth and Clebsch."

At the meeting of May 11 the following papers were read: By G. T. BENNETT, "Exhibition of a model of a deformable octahedron"; by J. W. NICHOLSON, "The scattering of light by a large conducting sphere."

THE International commission on the teaching of mathematics will hold its meeting this year at Milan September 18-20 under the presidency of Professor F. KLEIN.

THE fourth international congress of philosophy was held at

Bologna April 6-11 under the presidency of Professor F. ENRIQUES.

THE Italian mathematical society Mathesis has undertaken the preparation of an encyclopedia of elementary mathematics, primarily for the use of teachers in the secondary schools. The board of editors is composed of Professors L. BERZOLARI, G. VIVANTI, F. GERBALDI, R. BONOLA and E. VENERONI. The work will comprise 44 monographs, to be published in three volumes, the first of which will treat of analysis, the second of geometry, and the third of applications, history, and didactics.

THE provisional report of the national committee of fifteen on geometry has been completed and will be presented at the next meeting of the National educational association to be held at San Francisco, July 8 to 14.

A SPANISH mathematical society has been organized at Madrid, where its first meeting was held on April 5. J. ECHEGARAY was elected president. The society will publish a *Bulletin* which will be in charge of C. J. RUEDA, L. O. DE TOLEDO, A. KRAHE and J. R. PASTOR.

TO perpetuate the memory of the late Professor J. TANNERY, of the Ecole Normale Supérieure, a committee composed of the director of the school, M. Poayet; M. Liard, the rector of the University of Paris; Professors Poincaré and Darboux have issued an appeal to all who knew Professor Tannery and to others who are mindful of the work he accomplished to assist in having his portrait hung in the public hall of the Ecole Normale. Contributions should be sent to the treasurer, M. Paul Pupuy, secretary of the Ecole Normale, 45 rue d'Ulm, Paris V.

THE Smith's prize committee of Cambridge University has declared G. H. LIVENS, of Jesus College, first prize man, for his essay: "The influence of density upon the position of lines of emission and absorption in a gas spectrum," and W. E. H. BERWICK, of Clare College, second prize man for his essay: "On the theory of relative fields." The essays of C. G. DARWIN, Trinity College, S. LEES, St. John's, and A. W. H. THOMPSON, Trinity, received honorable mention.

THE following doctorates in mathematics were conferred by the University of Paris during the academic year 1909-1910:

L. ROY, "Recherches sur les propriétés thermo-mécaniques des corps solides." (doctorat d'état); M. HAAG, "Familles de Lamé, composées de surfaces égales, généralisation, applications" (doctorat d'état); Z. DE GEOCZE, "Quadrature des surfaces courbes" (doctorat d'université).

THE following university courses are announced for the summer of 1911:

UNIVERSITY OF CHICAGO. Summer quarter (June 19–September 1). — By Professor E. H. MOORE: Integral equations, four hours, first term; General analysis, five hours, first term. — By Professor J. W. YOUNG: Synthetic projective geometry, four hours. — By Professor J. W. A. YOUNG: Review of secondary mathematics, four hours. — By Professor G. A. BLISS: Elliptic integrals, four hours; Calculus of variations, four hours. — By Professor H. F. BLICHFELDT: Finite collineation groups, four hours. — By Professor A. C. LUNN: Analytic mechanics, four hours; Vector analysis, four hours.

COLUMBIA UNIVERSITY. Summer session (July 5–August 16). — By Professor H. S. WHITE: Theory of functions of a real variable. — By Professor EDWARD KASNER: Differential geometry. — By Professor W. B. FITE: Higher algebra.

INDIANA UNIVERSITY. Summer term (June 22 to September 1, 1911). — By Professor S. C. DAVISSON: Theory of surfaces, five hours. — By Professor D. A. ROTHROCK: Calculus of variations, five hours (first half-term); Advanced calculus, five hours; History of mathematics (first half-term). — By Professor U. S. HANNA: Ordinary differential equations, five hours; Surveying, five hours. — By Mr. K. P. WILLIAMS: Solid analytic geometry (first half-term).

THE following university courses are announced for the academic year 1911–1912:

COLUMBIA UNIVERSITY. — By Professor C. J. KEYSER: Modern theories in geometry, three hours; The principles of mathematics, three hours. — By Professor T. S. FISKE: Advanced calculus, introduction to the theory of functions of a real variable, three hours; Theory of functions of a complex variable, three hours. — By Professor F. N. COLE: Theory of groups, three hours; theory of invariants, three hours. — By

Professor JAMES MACLAY: Higher algebra, three hours; Elliptic functions, three hours. — By Professor D. E. SMITH: History of mathematics, three hours. — By Professor EDWARD KASNER: Differential equations, three hours, second half year; Dynamical geometry, three hours. — By Dr. N. J. LENNES: General theory of assemblages, three hours.

The mathematical colloquium will meet at intervals of about two weeks.

In October and November Professor J. HADAMARD, of the Collège de France, will give at Columbia University courses extending through four or five weeks as follows: Calculus of variations, Tuesday and Thursday afternoons from four to six; Partial differential equations of physics, Wednesday and Friday afternoons from four to six; four lectures on The definition of solutions of linear partial differential equations by boundary conditions, Recent application of certain mathematical theories to physical problems, Analysis situs, and Elementary solutions of partial differential equations and Green's functions, Saturday mornings from half past ten to half past twelve.

The Saturday lectures are open to all persons interested in mathematics.

UNIVERSITY OF ILLINOIS. — By Professor E. J. TOWNSEND: Theory of functions of a complex variable, three hours; Seminar in special topics in the theory of functions. — By Professor S. W. SHATTUCK: Differential equations and calculus of variations, three hours. — By Professor G. A. MILLER: Theory of numbers, three hours. — By Professor H. L. RIETZ: Theory of statistics, three hours. — By Professor C. H. SISAM: Solid analytic geometry, three hours. — By Professor J. B. SHAW: Theory of potential and related functions, three hours; Seminar course in general algebra. — By Professor A. EMCH: Projective geometry, three hours. — By Dr. A. R. CRATHORNE: Theory of functions of a real variable, three hours. — By Dr. R. L. BORGER: Invariants and higher plane curves, three hours. — By Dr. E. B. LYTLE: Teachers course in mathematics, two hours (second semester).

INDIANA UNIVERSITY. — By Professor S. C. DAVISSON: Theory of functions, three hours (a, w, s). — By Professor D. A. ROTHROCK: Contact transformations, two hours (a, w); Differential equations, three hours (a, w); Fourier series, three hours (s). — By Professor U. S. HANNA: Elliptic functions,

two hours (a, w, s); Advanced calculus, three hours (a, w, s). — By Professor R. D. CARMICHAEL: Linear differential equations, three hours (a, w, s); Difference equations, three hours (a, w, s). (*Note* Autumn, winter, spring terms are denoted by a, w, s .)

JOHNS HOPKINS UNIVERSITY. — By Professor F. MORLEY: Higher geometry three hours; Theory of functions, two hours; Vector analysis, two hours. — By Professor A. B. COBLE: Theory of groups, three hours; Theory of correspondences, three hours. — By Dr. A. COHEN: Elementary theory of functions, three hours; Differential equations, two hours; Differential geometry, two hours.

PRINCETON UNIVERSITY. — (In addition to the courses announced in the May BULLETIN.) By Professor O. VEBLER: Seminar in geometric group theory, both terms. — By Professor G. D. BIRKHOFF: Seminar in linear difference equations, both terms. — By Professor E. P. ADAMS: Analytic mechanics, three hours.

YALE UNIVERSITY. — By Professor J. PIERPONT: Theory of functions of a complex variable, two hours; Modern analytic geometry, two hours; Advanced differential equations, two hours; Theory of numbers, two hours. — By Professor P. F. SMITH: Differential geometry, two hours; Geometric analysis, one hour. — By Professor E. W. BROWN: Mechanics, two hours; Advanced calculus, three hours; Celestial mechanics, two hours. — By Professor W. R. LONGLEY: Calculus of variations, two hours. — By Dr. H. F. MACNEISH: Differential equations, one hour. — By Dr. G. M. CONWELL: Foundations of geometry, two hours. — By Dr. G. F. GUNDELFINGER: Analytic geometry, two hours. — By Dr. D. D. LEIB: Transformations of space, two hours. — By Dr. E. J. MILES: Advanced algebra, two hours. — By Mr. W. A. WILSON: Theory of functions of a real variable, two hours.

PROFESSOR D. HILBERT, of the University of Göttingen, has been elected corresponding member of the academy of sciences of Paris. Professor LORENTZ, of the University of Leyden, has been elected foreign member of the same academy.

PROFESSOR M. NOETHER, of the University of Erlangen, has been elected foreign associate of the Italian academy of sciences (the XL)

PROFESSOR L. SCHLESINGER, of the University of Budapest, has accepted a professorship of mathematics at the University of Giessen.

PROFESSOR MAX DEHN, of the University of Münster, has been appointed to an associate professorship of mathematics at the University of Kiel.

THE Adams prize has been awarded to Professor A. E. H. LOVE for his paper on "Certain problems of geodynamics."

THE University of Aberdeen has conferred the honorary degree of LL.D. on Major P. A. MACMAHON.

THE following members of the mathematical staff of the University of Moscow have resigned: Professor CERASKI, S. TSCHAPLIGIN, Professor B. MŁODZEJEWSKY, Dr. S. VINOGRADOW, Dr. A. VLASOV, Dr. A. VOLKOV, Dr. J. ZEGALKIN, Dr. A. POLJAKOV.

AT the meeting of the National academy of sciences held at Washington, April 20, Professor E. B. VAN VLECK, of the University of Wisconsin, was elected to membership; Professor V. VOLTERRA, of the University of Rome, was elected a foreign associate.

PROFESSOR H. Y. BENEDICT, of the University of Texas, has been made dean of the college of arts.

AT the Massachusetts Institute of Technology, associate professor E. B. WILSON has been promoted to a full professorship of mathematics.

PROFESSOR J. W. YOUNG, of the University of Kansas, has been appointed professor of mathematics in Dartmouth College.

PROFESSOR J. K. WHITTEMORE, of Harvard University, has been appointed professor of mathematics at Western Reserve University.

PROFESSOR F. L. GRIFFIN, of Williams College, has been appointed professor of mathematics at Reed College, Portland, Ore.

DR. E. J. MILES, of Cornell University, has been appointed instructor in mathematics in the Sheffield Scientific School of Yale University.

At the University of Illinois the following changes are announced: Dr. R. K. MORLEY, of the Worcester Polytechnic Institute, has been appointed instructor in mathematics; Mr. E. B. STOFFER and Mr. S. A. RAWLIN have been appointed assistants in mathematics. Dr. THOMAS BUCK has been appointed instructor in mathematics at the University of California.

MR. R. E. RICE, of the University of Chicago, has been appointed instructor in mathematics at the University of Missouri.

DR. H. H. MITCHELL, of Yale University, has accepted an instructorship in mathematics at the University of Pennsylvania.

DR. FRANK IRWIN, of Princeton University, has been appointed instructor in mathematics in the University of California.

At Columbia University Professor C. B. UPTON has been appointed secretary of Teachers College. Mr. L. L. DINES and Mr. G. H. GRAVES have been appointed instructors in mathematics.

PROFESSOR R. BONOLA, of the normal school at Rome, died May 16 at the age of 36 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ASHTON (C. H.). Die Heineschen O-Funktionen und ihre Anwendungen auf die elliptischen Funktionen. (Diss.) München, 1909. 8vo. 62 pp.

BAKER (A. L.). Quaternions as the result of algebraic operations. New York, Van Nostrand, 1911. 12mo. 100 pp. \$1.25

BAKER (R. P.). Problem of the angle-bisectors. Chicago, University of Chicago, 1911. 4to. 104 pp. \$1.00

BALL (W. W. R.). Ri-reazioni e problemi matematici dei tempi antichi e moderni. Bologna, Zanichelli, 1911. 8vo. 2+398 pp. L. 10.00

BRÜNS (M.). Zur Theorie der desmischen Flächen vierter Ordnung. (Diss.) Bonn, 1910. 8vo. 48 pp.

CARLBACH (J.). Lewi ben Gerson als Mathematiker. Ein Beitrag zur Geschichte der Mathematik bei den Juden. Berlin, Lamm, 1910. 8vo. 239 pp. Cl-th. M. 6.00

- BUCHER, neue, über Naturwissenschaften und Mathematik. Die Neuigkeiten des deutschen Buchhandels nach Wissenschaften geordnet. Leipzig, Hinrichs, 1911. 8vo. pp. 65-88. M. 0 30
- COURANT (R.). Ueber die Anwendung des Dirichletschen Prinzips auf die Probleme der konformen Abbildung. Göttingen, 1910. 8vo. 44 pp.
- DINGLER (H.). Die Grundlagen der angewandten Geometrie. Eine Untersuchung über den Zusammenhang zwischen Theorie und Erfahrung in den exakten Wissenschaften. Leipzig, Akademische Verlagsgesellschaft, 1911. 8vo. 8+160 pp. M. 6.00
- DÜKER (W.). Ueber Beziehungen der Strahlenkomplexe zweiten Grades zu den Flächen zweiter Ordnung. Rostock, 1910. 8vo. 80 pp
- FORSTER (J.). Allgemeine elementare Lösung des Problems von Fermat, $x^n + y^n = z^n$, $n > 2$, mittelst der "descente infinie." Wien, Holzhausen, 1911. 8vo. 10 pp. M. 0 50
- FROST (P.). An elementary treatise on curve tracing. 2d edition. London, Macmillan, 1911. 8vo. 10 s.
- HEFFTER (L.). Ueber Wesen, Wert, und Reiz der Mathematik. Rede. Kiel, Lipsius, 1911. 8vo. 14 pp. M 0 60
- HELLINGER (E.). Neue Begründung der Theorie quadratischer Formen von unendlichvielen Veränderlichen. Marburg, 1909. 4to. 62 pp.
- JANS (C.). Over Krommen von Clairaut, Boscovich en Playfair. Gent, Hoste, 1910.
- KLINGELHÖFFER (R.). Der Fermatsche Satz. Frankfurt, 1910. M. 2.00
- KOBER (H.). Konjugierte kinetische Brennpunkte. Breslau, 1910. 8vo. 77 pp.
- KOCH (W.). Beiträge zur affinen Geometrie der Flächen zweiten Grades. (Diss.) Kiel, 1910. 8vo. 51 pp.
- KOMMERELL (V.) und KOMMERELL (K.). Spezielle Flächen und Theorie der Strahlensysteme. (Sammlung Schubert, LXII.) Leipzig, Göschen, 1911. 8vo. 6+171 pp. Cloth. M. 4.80
- LEBON (E.). Sur le calcul avec une équation indéterminée des caractéristiques entrant dans une table de facteurs premiers des grands nombres. Paris, Société Philomatique de Paris, 1910. 8vo. 7 pp.
- LEFEBVRE (E.). Cours d'analyse infinitésimale de l'Ecole militaire. Deuxième partie. Livre I. Gand, Meyer, 1911. 8vo. 14+456 pp. Fr. 10 00
- METH (B.). Zur Erinnerung an Ernst Eduard Kummer als Lehrer. Berlin, Mathematischer Verein der Universität, 1910. M. 0.30
- MONTFORT (P.). Die Auflösung der numerischen Gleichungen nach Fourier. Leipzig, Teubner, 1910
- MUNTZ (C.). Zum Randwertproblem der partiellen Differentialgleichung der Minimalflächen. (Diss.) Berlin, 1910. 4to. 32 pp.
- PFEIFFER (F.). Das die Raumkurven begleitende Dreikant. (Diss.) Würzburg, 1910. 8vo. 35 pp.
- PRASAD (G.). A text-book of differential calculus. New York, Longmans, 1911. 8vo. 10+161 pp. \$1 50

- . A text-book of integral calculus. New York, Longmans, 1911. 8vo. 10+241 pp. \$1.50.
- PUSTAU (W.). Lösung des grossen Fermatschen Satzes. Frankfurt, Knauer, 1911. 8vo. 12 pp. M. 2.00
- RATSCHLÄGE und Erläuterungen für die Studierenden der Mathematik und Physik an der Universität Göttingen. Leipzig, Teubner, 1911. 8vo. 34 pp. M. 1.00
- RICHERT (P.). Die ganze rationale Funktion vierten Grades und ihre Kurven. Berlin, 1910. 4to. 19 pp.
- * PASCAL (E.). Lezioni di calcolo infinitesimale. (Manuali Hoepli.) Parte II. 3a edizione riveduta. Milano, Hoepli, 1911. 16mo. 8+330 pp.
- S. (F. R.). Calculus made easy. A very simplest introduction to those beautiful methods of reckoning which are generally called by the terrifying names of the differential and integral calculus. New York, Macmillan, 1911. 12mo. 8+178 pp. \$0.75
- SCHIEFFERS (G.). Lehrbuch der Mathematik für Studierende der Naturwissenschaften und der Technik. Einführung in die Differential- und Integralrechnung und in die analytische Geometrie. Zweite Auflage. Leipzig, Veit, 1911. 8vo. 8+732 pp. Cloth. M. 19.50
- SCHNEIDER (J. J.). Die Kugelfallprobe. Berlin-Charlottenburg, 1910. 8vo. 50 pp.
- SCHWANTKE (C.). Ueber den axiomatischen Aufbau einer Geometrie linearer Kugelsysteme. Marburg, 1910. 8vo. 42 pp.
- SÉRRET (J. A.). Cours de calcul différentiel et intégral. 6e édition. 2 volumes. Paris, Gauthier-Villars, 1911. 8vo. Tome 1er, 8+617 pp.; tome 2, 8+904 pp. Fr. 25.00
- STACKEL (P.). Geltung und Wirksamkeit der Mathematik. Karlsruhe, 1910. 8vo. 18 pp.
- STEINBACHER (F.). Abelsche Körper als Kreisteilungskörper. Berlin, 1910. 4to. 16 pp.
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- AMALDI (U.). See ENRIQUES (F.).
- AMODEO (F.). Aritmetica ed algebra, ad uso delle scuole medie di 2° grado. Napoli, Pierro, 1910. 16mo. 16+719 pp. L. 5.50
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- . Elementi di algebra. (Elementi di matematica.) Parte I del volume II. 2a edizione, migliorata. Napoli, Pierro, 1910. 16mo. 12+180 pp. L. 2.00
- . Elementi di matematica. Volume II, per la 2a classe normale. 5a edizione. Firenze, LeMonnier, 1911. 8vo. 185 pp. L. 1.50
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- ARZELÀ (C.). Trattato d'algebra elementare. (Collezione scolastica.) Terza edizione, completamente rifatta, 5a impressione. Firenze, Le Monnier, 1911. 16mo. 11+503 pp. L. 3.50
- BOSIO (C.). Nozioni di aritmetica, geometria e computisteria, corredate da molti esercizi orali e scritti. Torino, Paravia, 1910. 16mo. 272 pp. L. 1.00
- BOURLET (C.). Cours abrégé de géométrie avec de nombreux exercices théoriques et pratiques et des applications au dessin géométrique. 3e édition. Paris, Hachette, 1911. 16mo. 8+239 pp. Fr. 1.80
- . Pequeño curso de aritmética. Primera parte. Clases preparatorias. Versión castellana de la quinta edición francesa, por D. M. Ruiz. 1a edición. Paris, Hachette, 1911. 16mo. 136 pp. Fr. 1.50
- CHILD (J. M.). See LOCK (J. B.).
- COATES (J. V. N.). A first book of geometry. (First books of science.) London, Macmillan, 1911. 8vo. 10+142 pp. 1s. 6d.
- COLART (E.). Compléments d'algèbre élémentaire à l'usage de l'enseignement moyen et des candidats aux écoles spéciales. Quatrième édition. Bruxelles, De Boeck, 1911. 8vo. 188 pp. Fr. 3.50
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- AUE (J.). Zur Berechnung der Spannungen in gekrümmten Stäben unter Anwendung der optischen Methode. Jena, 1910. 8vo. 49 pp
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- BOUSSINESQ (J.). Sur les principes de la mécanique et sur leur applicabilité à des phénomènes qui semblent mettre en défaut certains d'entre eux. 1910. 4to. 38 pp. Fr. 2.00
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- CHURCH (A. E.) and BARTLETT (G. M.). Elements of descriptive geometry; with applications to spherical and isometric projections, shades and shadows, and perspective. New York, American Book Co., 1911. 8vo. 286 pp. \$2.25
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- KUSTER (F. W.). Logarithmische Rechentafeln für Chemiker, Pharmazeuten, Mediziner, und Physiker. Leipzig, Veit, 1911. 8vo. 107 pp. Cloth. M. 2.40
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- MAREC (E.). Les enroulements industriels des machines à courant continu et à courants alternatifs. Avec une préface de P. Janet. 1911. 8vo. 10+240 pp. Fr. 9.00
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- MORLEY (A.). Mechanics for engineers. 3d edition. New York, Longmans, 1911. 12mo. 11+290 pp. \$1.20
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- PLANCK (M.). Vorlesungen über Thermodynamik. Dritte erweiterte Auflage. Leipzig, Veit, 1911. 8vo. 8+288 pp. Cloth. M. 7.50
- RICHARZ (F.) und NEUMANN (E.). Ueber ein Mass des Verdachtes auf systematische Fehler. Marburg, 1909. 4to. 21 pp.
- SCHILLING (K.). Ueber die Elektrizitätsleitung in festen Körpern und die Minimalwerte des Widerstandes. Freiburg, 1910. 8vo. 71 pp.
- SMITH (A.). Stresses in simple framed structures. A text-book to accompany exercises in the computation of the axial stresses in the members of load bearing frames. West Lafayette, Ind., Smith, 1911. 8vo. 190 pp. \$2.50
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- TUTTON (A. E. H.). Crystallography and practical measurements. London, Macmillan, 1911. 8vo. 960 pp. 30s
- VACQUANT (C.). See BRIOT (C.).
- WOODWARD (C. J.). A B C of five-figure logarithms and tables for chemists, including electro-chemical equivalents, analytical factors, gas reduction tables, and other tables useful in chemical laboratories. New York, Spon, 1911. 12mo. 75 pp. \$1 00
- YULE (G. U.). An introduction to the theory of statistics. London, Griffin, 1911. 8vo. 390 pp. 10s. 6d.

THE CHICAGO MEETING OF THE AMERICAN
MATHEMATICAL SOCIETY.

AT the invitation of the Chicago Section the April meeting of the Society was held this year at the University of Chicago, on Friday and Saturday, the 28th and 29th of the month. This was in many ways a remarkable occasion. It was the first regular meeting of the Society except the summer meetings, to be held away from New York City since its founding there in 1888. And it was arranged that this reunion of the eastern and western members should be especially marked by the delivery of President Bôcher's retiring address, which was postponed for that purpose from the annual meeting of the Society in the preceding December. As was under these circumstances to be expected, the meeting was in every way a most successful and inspiring one. Fifty-three papers filled to overflowing the programme of the four sessions. The attendance numbered 115 persons, including the following 88 members of the Society :

Professor G. N. Armstrong, Professor H. Bateman, Professor W. H. Bates, Professor G. N. Bauer, Professor C. H. Beckett, Professor W. W. Beman, Professor G. A. Bliss, Professor Maxime Bôcher, Dr. R. L. Börger, Professor J. W. Bradshaw, Professor W. C. Brenke, Mr. Daniel Buchanan, Dr. H. E. Buchanan, Dr. Thomas Buck, Dr. H. T. Burgess, Professor W. H. Bussey, Professor D. F. Campbell, Professor H. E. Cobb, Professor F. N. Cole, Dr. A. R. Crathorne, Professor D. R. Curtiss, Professor E. W. Davis, Professor S. C. Davisson, Professor L. E. Dickson, Mr. L. L. Dines, Professor L. W. Dowling, Dr. Arnold Dresden, Professor L. P. Eisenhart, Professor Arnold Emch, Professor Peter Field, Professor H. B. Fine, Professor W. B. Ford, Professor J. W. Glover, Professor A. G. Hall, Professor F. T. H'Doubler, Professor T. F. Holgate, Professor O. D. Kellogg, Professor A. M. Kenyon, Professor Kurt Laves, Dr. N. J. Lennes, Professor A. C. Lunn, Dr. E. B. Lytle, Professor Malcolm McNeill, Dr. W. D. MacMillan, Professor Max Mason, Mr. R. M. Mathews, Professor G. A. Miller, Mr. W. J. Montgomery, Professor C. N. Moore, Professor E. H. Moore, Dr. R. L. Moore, Professor J. C. Morehead, Professor F. R. Moulton, Professor G. W.

Myers, Dr. L. I. Neikirk, Professor W. F. Osgood, Professor G. P. Paine, Professor Alexander Pell, Dr. Anna J. Pell, Professor W. S. Pemberton, Professor A. D. Pitcher, Professor H. L. Rietz, Professor W. J. Risley, Professor W. H. Roever, Mr. R. E. Roet, Professor D. A. Rothrock, Professor T. R. Running, Mr. J. M. Rysgaard, Miss I. M. Schottenfels, Mr. A. R. Schweitzer, Professor J. B. Shaw, Professor Mary E. Sinclair, Professor C. H. Sisam, Professor E. B. Skinner, Professor H. E. Slaughter, Professor C. S. Slichter, Professor P. F. Smith, Mr. V. M. Spunar, Professor C. E. Stromquist, Professor E. J. Townsend, Professor A. L. Underhill, Professor E. B. Van Vleet, Dr. G. E. Wahlin, Professor C. B. Williams, Professor R. E. Wilson, Professor A. E. Young, Professor J. W. Young, Professor Alexander Ziwet.

President H. B. Fine occupied the chair, being relieved by Professor G. A. Miller and Vice-President G. A. Bliss. The Council announced the election of the following persons to membership in the Society: Professor H. Bateman, Bryn Mawr College; Mr. Samuel Beatty, University of Toronto; Professor J. H. Griffith, University of Michigan; Mr. E. J. Moulton, Harvard University; Mr. George Spitzer, Purdue University; Professor C. J. West, Ohio State University. Eleven applications for membership in the Society were received.

Professor L. E. Dickson was reelected member of the Editorial Committee of the *Transactions* for the term of three years. The President was authorized to appoint a committee to arrange for the summer meeting and colloquium of the Society to be held at the University of Wisconsin in 1913. Provision was also made for a committee to prepare a list of nominations of officers for the annual election in December. It was decided to hold the annual meeting this year in New York City.

On Friday evening seventy-three members gathered at the dinner at the Quadrangle Club, where luncheon was also served on both days of the meeting. These and other lesser gatherings afforded very agreeable opportunities for renewing and widening personal acquaintance and comparing scientific and other notes. The meeting brought together members from Massachusetts to Wyoming and from Minnesota to Missouri and contributed greatly to strengthen the bonds of the Society's common interest.

The Presidential Address of Professor Bôcher, "Charles

Sturm's published and unpublished work on differential and algebraic equations," was delivered at the opening of the afternoon session on Friday. The address will appear in an early number of the next volume of the BULLETIN.

The following papers were read at the Chicago meeting :

(1) Mr. DANIEL BUCHANAN : " A class of periodic solutions of the problem of three bodies, two of equal mass, the third moving in a straight line."

(2) Dr. H. E. BUCHANAN : " An expansion of elliptic functions with applications."

(3) Professor D. R. CURTISS : " Relations between the Gramian, the Wronskian, and a third determinant connected with the problem of linear dependence."

(4) Mr. L. L. DINES : " On the representation of resultants of n polynomials in one variable."

(5) Mr. L. L. DINES : " On the solution of three equations for the three variables in terms of others."

(6) Dr. W. D. MACMILLAN : " A reduction of a system of power series to an equivalent system of polynomials."

(7) Dr. W. D. MACMILLAN : " A method for finding the solutions of a set of analytic functions in the neighborhood of a branch point."

(8) Dr. R. L. MOORE : " On the transformation of double integrals."

(9) Professor MAXIME BÔCHER : Presidential Address — " Charles Sturm's published and unpublished work on differential and algebraic equations."

(10) Professor L. P. EISENHART : " A fundamental parametric representation of space curves."

(11) Professor A. E. YOUNG : " On certain orthogonal systems of lines and the problem of determining surfaces referred to them."

(12) Professor ARNOLD EMCH : " The differential equation of curves of normal stresses in a plane field."

(13) Dr. A. B. FRIZELL : " A set of postulates for well ordered types."

(14) Professor C. J. KEYSER : " Sensuous representation of paths that lead from the inside to the outside of an ordinary sphere in point four-space without penetrating the surface of the sphere."

(15) Professor EDWARD KASNER : " The subdivisions of curvilinear angles."

(16) Mr. R. D. CARMICHAEL: "The general theory of linear q -difference equations."

(17) Mr. R. D. CARMICHAEL: "Note on multiply perfect numbers."

(18) Professor G. A. MILLER: "Isomorphisms of a group whose order is ϵ power of a prime."

(19) Professor R. G. D. RICHARDSON: "Theorems of oscillation for two self-adjoint linear differential equations of the second order with two parameters (second paper)."

(20) Professor J. B. SHAW: "Quaternion functions of three parameters."

(21) Dr. J. E. ROWE: "The combinants of two binary cubics and their geometrical interpretation on the rational cubic curve."

(22) Professor U. G. MITCHELL: "Geometry and collineation groups of the finite projective plane $PG(2, 2^n)$."

(23) Dr. G. E. WAHLIN: "The decomposition of rational primes into ideal prime factors in the field $k(\sqrt[m]{m})$."

(24) Professor L. C. KARPINSKI: "An Italian Algebra of the fifteenth century."

(25) Professor C. H. SISAM: "On hyperconical connexes in space of r dimensions."

(26) Mr. R. E. ROOT: "Iterated limits of functions on an abstract range."

(27) Professor E. B. VAN VLECK: "On the generalization of a theorem of Poincaré."

(28) Professor E. B. VAN VLECK: "On the classification of collineations."

(29) Mr. A. R. SCHWEITZER: "On the philosophy of Grassmann's extensive algebra."

(30) Mr. A. E. SCHWEITZER: "On the 'working hypothesis' in the logic of mathematics."

(31) Professor W. B. FORD: "A set of sufficient conditions that a function may have an asymptotic representation in a given region."

(32) Mr. W. J. MONTGOMERY: "The classification of twisted curves of the fifth order."

(33) Professor WILLIAM MARSHALL: "On Hill's differential equation in the theory of perturbations."

(34) Professor H. BATEMAN: "The fundamental equations of the theory of electrons and the infinitesimal transformation of an electromagnetic field into itself."

(35) Dr. N. J. LENNES: "Curves and surfaces in analysis situs."

(36) Dr. N. J. LENNES: "Extension and application of a theorem of Ascoli."

(37) Dr. L. I. NEIKIRK: "Substitution groups of an infinite degree and their related functions."

(38) Professor JAMES MACLAY: "Parabolic curves."

(39) Mr. J. A. NYBERG: "Projective differential geometry of rational cubic curves."

(40) Mr. E. B. STOFFER: "Invariants of linear differential equations with applications to ruled surfaces in five-dimensional space."

(41) Dr. W. D. MACMILLAN: "A general existence theorem for periodic solutions of differential equations of a certain type."

(42) Dr. A. R. CRATHORNE: "The catenary with variable end points."

(43) Professor F. R. MOULTON: "Periodic orbits of superior planets."

(44) Professor F. R. MOULTON: "On the curves defined by certain differential equations."

(45) Professor F. H. SAFFORD: "An identical transformation of the elliptic element in the Weierstrass form."

(46) Professor W. H. ROEVER: "Southerly deviation of falling bodies (third paper)."

(47) Professor C. N. MOORE: "Convergence factors in double series."

(48) Professor L. E. DICKSON: "On negative discriminants for which there is a single class of positive primitive binary quadratic forms."

(49) Professor L. E. DICKSON: "On Fermat's 'descente infinie.'"

(50) Professor L. E. DICKSON: "On perfect numbers and Bernoullian numbers."

(51) Professor O. E. GLENN: "On expressing a quantic in terms of assigned powers of a given quantic."

(52) Professor G. R. CLEMENTS: "Implicit functions defined in the neighborhood of a point where the Jacobian determinant is zero."

(53) Professor R. W. MARRIOTT: "Determination of the groups of isomorphisms of the groups of order p^4 ."

Mr. Nyberg and Mr. Stoffer were introduced by Professor

- Wilczynski. Professor Marriott's paper was communicated to the Society through Professor Hallett. The papers of Dr. Frizell, Professor Keyser, Professor Kasner, Mr. Carmichael, Professor Richardson, Dr. Rowe, Professor Mitchell, Professor Karpinski, Mr. Schweitzer, Professor Marshall, Professor Mac-lay, Professor Safford, Professor Roeber, Professor Dickson (first paper), Professor Glenn, Professor Clements, and Professor Marriott were read by title.

Professor Curiss's paper was published in the June BULLETIN. Mr. Root's paper and Professor Dickson's first paper are contained in the present number of the BULLETIN. Professor Dickson's third paper appeared in the May number of the *American Mathematical Monthly*. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. At the Minneapolis meeting of the Chicago Section Dr. W. D. Macmillan in his paper on "An integrable case in the problem of three bodies" discussed the periodic motion of three bodies, two of equal mass moving in a circle about their center of gravity, the third an infinitesimal moving on a straight line drawn through the center of gravity perpendicular to the plane of motion of the other two. The paper by Mr. Buchanan shows that periodic solutions exist when the third body is considered finite. The solutions are constructed as power series in the mass of the third body, the coefficients being power series in the initial projection from the plane of motion of the other two. The paper shows that periodic solutions exist also when the equal bodies move in ellipses and the third is infinitesimal.

2. The differential equation

$$\frac{\partial^2 x}{\partial t^2} = -a\{(1 + \lambda)x - 2\lambda x^3\}$$

defines x as a doubly periodic function of t . In the first section of his paper Dr. Buchanan shows how to obtain x as a power series in λ , the coefficients of the series being simply periodic functions of t . It is then shown how to use this series in the integration of certain types of linear homogeneous differential equations with doubly periodic coefficients. Lamé's equation is used as an example.

The second section is a discussion of the motion of a heavy particle on the surface of an anchor ring, using methods similar to those discussed in the first section.

4. In his first paper Mr. Dines considers conditions for the existence of a common root of n polynomials in one variable. F. Meyer has shown (*Jahresbericht der Deutschen Mathematiker-Vereinigung*, volume 16) that necessary and sufficient conditions can be stated in terms of the vanishing of $n - 1$ functions of the coefficients. These functions are obtained as remainders in a process of successive divisions which is a generalization of the euclidean algorithm for the determination of the greatest common divisor of two polynomials. Foethke, in his dissertation (Königsberg, 1907), has shown by an analysis of the Meyer process that under certain restrictions upon the coefficients of the polynomials, necessary and sufficient conditions can be stated in terms of the vanishing of $n - 1$ determinants. Mr. Dines, by a different method, more like the Sylvester method for determining the resultant of two polynomials, shows that necessary and sufficient conditions can always be stated in terms of the vanishing of $n - 1$ determinants. If a certain restriction be made involving the coefficients of only two of the polynomials, these $n - 1$ determinant resultants can be formed according to a very simple law, their elements being coefficients of the polynomials. If this restricting condition is not satisfied, some of the resultants are complicated, the elements being themselves determinants. In certain cases these more complicated determinants can be reduced to simple ones, and it is possible that this is always true.

5. The second paper of Mr. Dines deals with the solutions of the three equations

$$\begin{aligned}\phi(u, v, w; x_1, \dots, x_p) &= 0, & \psi(u, v, w; x_1, \dots, x_p) &= 0, \\ \chi(u, v, w; x_1, \dots, x_p) &= 0\end{aligned}$$

for u, v , and w in terms of the x 's, where ϕ , ψ , and χ are analytic functions of all their variables in a certain region. If the functional determinant $D(\phi, \psi, \chi)/D(u, v, w)$ is different from zero at a point $(u^0, v^0, w^0; x_1^0, \dots, x_p^0)$ which satisfies the given equations, then as is well known, there is a unique set of analytic functions $u(x_1, \dots, x_p)$, $v(x_1, \dots, x_p)$, $w(x_1, \dots, x_p)$

which reduce to u^0, v^0, w^0 for $x_i = x_i^0$, and which satisfy the given equations identically in the x 's. If the functional determinant vanishes at a point which satisfies the given equations, for simplicity say the origin $(0, 0, 0; 0, \dots, 0)$, there are several sets of solutions $u(x_1, \dots, x_p), v(x_1, \dots, x_p), w(x_1, \dots, x_p)$, which converge to zero with the x 's. The number and to some extent the character of these solutions are considered. It is found that the number is in general lmn , where l is the degree of the lowest term of ϕ remaining when the x 's are put equal to zero, m is the degree of the lowest term in ψ , and n the degree of the lowest term in χ under the same conditions. The solutions for any one of the variables u, v, w can be found as the solutions of a polynomial of degree lmn in this variable, with the coefficient of the highest power unity, and the coefficients of the other terms power series in the x 's vanishing with the x 's. This is a generalization of the method given by the Weierstrass factor theorem for obtaining the solutions of a single equation with a single dependent variable, and of the method given by Professor Bliss for obtaining the solutions of two equations in two dependent variables. In a certain neighborhood of the origin, excluding certain point sets of lower dimension than the neighborhood along which some of the lmn sets of solutions coincide, the solutions are found to be analytic.

6. At the Minneapolis meeting of the Chicago Section Dr. MacMillan showed that a system of two power series in two dependent variables could be reduced to a system of two polynomials. In the present paper he shows that a system of any number of power series in the same number of dependent variables and of any number of independent variables can be reduced to an equivalent system of polynomials, provided the power series vanish at the origin and provided also there is a certain determinant distinct from zero. The word "equivalent" is to be understood as meaning that the power series set equal to zero admit the same solutions in the neighborhood of the origin as do the polynomials set equal to zero, and conversely. By means of this reduction he shows that the system has κ solutions in the neighborhood of the origin, κ being the product of the orders of the power series.

7. In his second paper Dr. Macmillan shows that if $x_1, \dots, x_n$ are defined implicitly as functions of parameters $\mu_1, \dots, \mu_p$

by means of the equations $f_i(x_1, \dots, x_n; \mu_1, \dots, \mu_p) = 0$ ($i = 1, \dots, n$), where the f_i are analytic in $x_1, \dots, x_n, \mu_1, \dots, \mu_p$, and regular for $x_1 = \dots = \mu_p = 0$, and if the terms of lowest degree in the expansions of $f_i(x_1, \dots, x_n; 0, \dots, 0)$ are of degree d_i , then the system of equations $f_i = 0$ admits $\prod_{i=1}^n d_i$ solutions for $x_1, \dots, x_n$ which vanish with $\mu_1, \dots, \mu_p$.

If there is but one parameter μ , these solutions are expansible as power series in integral or fractional powers of μ . In the present paper Dr. MacMillan shows how Newton's polygon can be used for finding the forms of the solutions. The first terms in the expansions of the solutions are determined by the solutions of sets of algebraic equations of degree not exceeding d_i . The terms beyond the first can be obtained by the solution of a set of equations $g_i(\xi_1, \dots, \xi_r; \mu^k) = 0$ ($i = 1, \dots, n$), whose functional determinant is in general distinct from zero.

If there are more parameters than one, say $\mu_1, \dots, \mu_p$, the solutions are not, in general, expansible as power series in integral or fractional powers of $\mu_1, \dots, \mu_p$.

8. At the last meeting of the Chicago Section Dr. Moore reported on a theorem whose hypothesis may, with some modifications, be stated as follows:

G is a limited point set in space of n dimensions. G_0 is a subset of G of measure 0. $G_{1n}, G_{2n}, \dots, G_{nn}$ are, for each value of the positive integer n , non-overlapping subsets of G whose interior measures are denoted by $e_{1n}, e_{2n}, \dots, e_{nn}$ respectively. $r_{1n}, r_{2n}, \dots, r_{nn}, r'_{1n}, r'_{2n}, \dots, r'_{nn}$ are real numbers whose totality taken for all values of n is a bounded set of values.

For any given point P and given value of n , i_{Pn} is to have meaning only if there exists a positive integer i ($i \leq n$) such that G_{in} contains the point P . In case such value of i exists it is to be designated by the symbol i_{Pn} . If P is a point of G , not belonging to G_0 , and such that for an infinite sequence of values of n , i_{Pn} exists (as above defined), then if n is confined to this sequence $\lim_{n \rightarrow \infty} (r'_{i_{Pn}n} - r_{i_{Pn}n}) = 0$.

Finally $\lim_{n \rightarrow \infty} \sum_{i=1}^n r_{in} e_{in}$ exists.

It was shown that if the above hypothesis is satisfied then

the following conclusion results:

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n r'_{in} e_{ir} \text{ exists and equals } \lim_{n \rightarrow \infty} \sum_{i=1}^n r_{in} e_{in}.$$

Dr. Moore proposes to apply this theorem to the transformation of double integrals.

10. In 1848 Serret proposed the problem of solving the differential equation $ds^2 = dx^2 + dy^2 + dz^2$, where s, x, y, z are functions of a single parameter, and he obtained a general method of solution. Darboux has discussed the same problem on two occasions, but it remained for Montcheuil, in 1905, to obtain a simple explicit solution, however, as a by-product of an entirely different problem. This solution is

$$\begin{aligned} x &= \phi - u\phi' + \psi', & iy &= \phi - u\phi' - \psi', & z &= \phi' + u\psi' - \psi, \\ & & s &= \phi' - u\psi' + \psi, \end{aligned}$$

where ϕ and ψ are arbitrary functions of u and the primes denote differentiation with respect to u . Professor Eisenhart shows that for the general space curve such a parameter u can be obtained in two ways, the exceptions being that for minimal curves and curves in an isotropic plane there is only one such parameter and that both parameters u are constant for straight lines. When the equations are written in this form, the determination of curves congruent in the euclidean sense reduces to the integration of a Schwarzian equation $\{u, s\} = \phi(s)$; consequently this investigation has close contact with recent work by Study. Curves of zero curvature and curves on the sphere of zero radius are readily studied when the above form of equations is used. One has also at once expressions for minimal curves in 4-space analogous to the Weierstrass formulas for minimal curves in 3-space. It is significant that a necessary and sufficient condition that a curve be algebraic is that x, y, z be algebraic functions of u . When it is desirable to deal entirely with real curves, the equations can readily be put in a more convenient form.

11. In a recent paper (*American Journal of Mathematics*, volume 32, No. 1), Professor Young discussed the problem of the spherical representation from the standpoint of the point coordinates, making use of the fundamental equations in

the Bonnet form. The present paper in many respects is a generalization of that. There the lines of reference were assumed to be lines of curvature on the surfaces, a condition which simplified considerably the equations involved and limited the results of solvable cases to particular surfaces. In the present paper, assuming in the first place that the lines of reference are merely orthogonal, a classification of certain such lines is made, based upon the fact that if these are chosen as lines of reference the most important of the fundamental equations, a Laplace equation, is integrable. These lines, which appear upon all surfaces, are called invariant lines as they are defined by equations which require that an invariant of a Laplace equation which belongs to the sequence formed from the given Laplace equation shall vanish.

A second classification of orthogonal lines, principal lines, as they are called, has been made, and the basis for this classification is the fact that the determination of such a set of lines on a surface makes possible the determination of another by algebraic methods from the known functions of the first. The second surface is an associate of the former, and hence its determination makes possible a deformation. Principal lines may or may not be invariant lines. Lastly, Professor Young has discovered a method for determining whether a surface is isothermic or not by algebraic methods requiring no quadrature and no reference to lines of curvature. This makes possible the extension of a very important theorem for the determination of a sequence of isothermic surfaces from any given one, so that the lines of reference need no longer be lines of curvature but may be any orthogonal system of lines whatsoever.

12. The stresses around a point (x_1, y_1) of a plane field are determined if the stresses acting on any two sections through the point are known. Assuming any analytic curve $y - y_1 = f(x - x_1)$ referred to cartesian coordinates, the stress P at the point (x_1, y_1) acting on a section tangent to the curve will be an analytic function of the coordinates and the slope

$$(1) \quad P = \phi \left(x_1, y_1, \frac{dy}{dx} \right).$$

Designating the slope of P by dy'/dx' it is known that the directions of corresponding sections and stresses form an involu-

toric pencil, so that

$$(2) \quad \frac{dy'}{dx'} = \frac{a \cdot dy/dx + b}{c \cdot dy/dx - a'}$$

where a , b , c are functions of x_1 and y_1 , determined by the conditions of the field.

The problem of Professor Emch's paper is to determine those curves for which corresponding tangent sections and stresses are normal and which, when considered as perfectly flexible but inelastic under the sole action of these stresses, remain in equilibrium. Writing $dy/dx = p$, the differential equation of this curve is found to be

$$(3) \quad \frac{dp}{dx}(1 + p^2)^{-\frac{1}{2}} = c \cdot \phi(x, y, p).$$

As an example, the field represented by the hydraulic pressures in a vertical plane through a fluid is worked out. Choosing a cartesian system in this plane with a horizontal and vertical axis, it is found that in this case the curves may be represented by elliptic integrals of the first and second kind. The differential equation for the orthogonal trajectories is obtained by replacing in (3) p in terms of $p' = dy'/dx'$ from (2), i. e., by putting $p' = (ap + b)/(cp - a)$.

13. In Cantor's process of definition the higher well ordered types are inextricably connected with the alephs by the notion of Abbildung. The alephs having proved unmanageable, it seems desirable to reject this idea and proceed simply according to the familiar postulates of arrangement. For example, in the second class, if we write $\alpha' = \omega^\alpha$, $\alpha'' = \omega^{\alpha'}$, etc., no one undertakes to form a working concept of the set of symbols between ω''' and $\omega^{\Gamma V}$ by counting successive ω -series and assigning to each a limit symbol as starting point for a new one. Instead we insert between ω'' and ω''' the set ω^α , where α runs through the previously defined set from ω'' to ω''' , and then between successive numbers of this new set interpolate in like manner other sets by the rules which yielded the lower symbols. Dr. Frizell makes use of this procedure to obtain a postulational definition of the second and higher classes of ordinal types.

14. By means of a transformation correlating the points of four-dimensional space S_4 with the spheres of ordinary space

S_3 , the spheres of real center and pure imaginary radius being included, Professor Keyser's paper shows how to render evident to spatial intuition the existence of point paths in S_4 that lead from the inside to the outside of an ordinary sphere without penetrating the surface and indicates how to make a simple physical model composed of spheres enabling one to distinguish by sight or touch the paths in question from those involving penetration of the surface.

15. Stated in the simplest case, that of bisection, the problem discussed by Professor Kasner is as follows: Given an angle whose sides a and b are analytic arcs, to construct through the vertex an arc c so that (Schwarz) reflexion in c shall interchange the sides. The general problem is fundamental in conformal geometry. For angles incommensurable with respect to π , solutions always exist; but for commensurable angles the problem is possible only when certain invariants vanish.

16. By means of a transformation of the form $z = (m_1x + m_2)/(\mu_1x + \mu_2)$, the system of functional equations

$$(1) \quad H_i\left(\frac{az+b}{cz+d}\right) = \sum_{j=1}^n \lambda_{ij}(z) H_j(z) \quad (i = 1, \dots, n)$$

in the n unknown functions $H_1(z), \dots, H_n(z)$ may be transformed into the system of difference equations

$$(2) \quad G_i(x+1) = \sum_{j=1}^n \bar{\lambda}_{ij}(x) G_j(x) \quad (i = 1, \dots, n)$$

or into the system of q -difference equations

$$(3) \quad G_i(qx) = \sum_{j=1}^n \bar{\lambda}_{ij}(x) G_j(x) \quad (i = 1, \dots, n)$$

according as the substitution $z' = (az+b)/(cz+d)$ has one or two double points. The essential properties of the solutions of (2) are known. (See the author's thesis and a paper by Professor Birkhoff in the *Transactions*, April, 1911, where references to the earlier literature are given.)

In the present paper, Mr. Carmichael proves the existence and obtains the fundamental properties of the solutions of a system of q -difference equations (3) for the case $|q| \neq 1$. In

many respects the theory is analogous to that of system (2), and the two theories interact in such a way that each throws light on the other. There is a fundamental system of solutions of simple character at zero and one of simple character at infinity. A study of the relations between these two systems of solutions gives rise to a theory analogous to the remarkable Birkhoff characterization of the solutions of a system of difference equations.

17. If the sum of all the divisors of an integer N is mN , where m is an integer, N is said to be a multiply perfect number of multiplicity m . Forty-one multiply perfect numbers are known; the most of them have been discovered by Euclid, Mersenne, Fermat, Descartes, Lucas, Lehmer, and Carmichael. It appears that the only rules heretofore known for determining these numbers, when the multiplicity is greater than 2, are those of Descartes (*Oeuvres* 2, pages 427–429); and even these are effective in only a few cases. After a short historical account of these numbers, Mr. Carmichael gives some simple rules which may be applied to known multiply perfect numbers in order to find others. The effectiveness of the rules in a favorable case is seen in the fact that if they are applied to a certain number given by Lehmer they lead at once to thirteen other multiply perfect numbers, all of which are new. The paper closes with a table of multiply perfect numbers. This credits to the discoverer each of the forty-one numbers which have previously been published. It contains in addition forty-two numbers which are new.

18. After observing that the group of isomorphisms of a group of order p^m cannot involve any operator whose order is divisible by p^m , Professor Miller determines all the groups of order p^m whose groups of isomorphisms involve operators whose orders are either p^{m-1} or p^{m-2} , p being any prime number. When p is odd, the cyclic group of order p^m is the only possible group of this order whose group of isomorphisms involves operators of order p^{m-1} . When $p = 2$, the group of isomorphisms of the cyclic group of order p^m involves no operator whose order exceeds p^{m-2} , but both the dihedral and the dicyclic group of order p^m have operators of order p^{m-1} in their groups of isomorphisms. Hence there are exactly two groups of order 2^m , $m > 3$, which have operators of order 2^{m-1} in their groups of isomorphisms.

A necessary and sufficient condition that the group of isomorphisms of a group of order p^m , $p > 2$ and $m > 4$, involves operators of order p^{m-2} is that the group itself involves operators of order p^{m-1} . Hence there are two and only two groups of order p^m , $p > 2$ and $m > 4$, which have operators of order p^{m-2} in their groups of isomorphisms without having also operators of order p^{m-1} in these groups. When $p = 2$ and $m > 5$, there are 13 groups of order p^m whose groups of isomorphisms involve operators of order p^{m-2} . An important theorem which was proved incidentally may be stated as follows:

If a group of order p^m , $p > 2$ and $m > 4$, contains no operator of order p^{m-1} , the number of its subgroups of order p^4 which do not include any operator of order p^3 is of the form $1 + kp$.

In particular, if a group of order p^m , $p > 2$, and $m > 4$, does not contain any operator of order p^{m-1} , it contains an invariant subgroup of order p^4 which does not involve any operator of order p^3 .

19. As an extension of the results reported to the Society in December 1910, Professor Richardson has derived necessary and sufficient conditions for the existence of solutions $u(x)$, $v(y)$, of the equations

$$(p_1 u')' + q_1 u + (\lambda A_{11} + \mu A_{12})u = 0,$$

$$(p_2 v')' + q_2 v + (\lambda A_{21} + \mu A_{22})v = 0 \quad (p_1(x) > 0, p_2(y) > 0),$$

which satisfy the boundary conditions $u(a_1) = u(b_1) = 0$, $v(a_2) = v(b_2) = 0$ and oscillate m and n times respectively. The problem is divided into six cases corresponding to different forms of the functions A_{11} , A_{12} , A_{21} , A_{22} . In two of these cases there always exist solutions; in the other four the necessary and sufficient conditions are nearly identical. For example, when $A_{12} \geq 0$, $A_{22} \geq 0$, A_{11} , A_{21} arbitrary, it is necessary that

$$\min \frac{A_{11}}{A_{12}} \geq \min \frac{A_{21}}{A_{22}}, \quad \max \frac{A_{11}}{A_{12}} \geq \max \frac{A_{21}}{A_{22}}$$

or that

$$\min \frac{A_{11}}{A_{12}} \leq \min \frac{A_{21}}{A_{22}}, \quad \max \frac{A_{11}}{A_{12}} \leq \max \frac{A_{21}}{A_{22}}.$$

When the equality sign is omitted these conditions are sufficient. The oscillation theorem for the equation

$$(py')' + qy + \lambda Ay = 0 \quad (p(x) > 0, y(0) = y(1) = 0),$$

where $q(x)$ is positive in at least a part of the interval and $A(x)$ takes both signs, may be stated as follows: The necessary and sufficient condition that there exist a solution which oscillates n times in the interval is that $n \geq n_1$, where n_1 is the number of zeros within the interval 0, 1 of that solution of $(py)' + qy = 0$ which vanishes at $x = 0$.

20. In Professor Shaw's paper the quaternion q is considered as a function of three scalar parameters u_1, u_2, u_3 ,

$$q = f(u_1, u_2, u_3).$$

The properties of the related operator

$$\Delta = \left(A_3 \cdot n q_2 q_3 \cdot \frac{\partial}{\partial u_1} + A_3 \cdot n q_3 q_1 \cdot \frac{\partial}{\partial u_2} + A_3 \cdot n q_1 q_2 \cdot \frac{\partial}{\partial u_3} \right) / \tau n^2$$

and of

$$\delta = \left(A_2 \cdot q_2 q_3 \cdot \frac{\partial}{\partial u_1} + A_2 \cdot q_3 q_1 \cdot \frac{\partial}{\partial u_2} + A_2 \cdot q_1 q_2 \cdot \frac{\partial}{\partial u_3} \right) / \tau n$$

are considered. The differential parameters for a function $\psi(u_1, u_2, u_3)$, dependent for its value on the same scalars as q , are expressed in terms of δ . The differential geometry of a curvilinear three-dimensional space in a four-dimensional flat space furnishes an application. The paper is a first generalization of that read at Minneapolis in December.

21. In Dr. Rowe's paper the equations of a rational cubic curve R^3 are written

$$(1) \quad x_i = a_i t^3 + 3b_i t^2 + 3c_i t + d_i, \quad (i = 0, 1, 2).$$

Two lines $(\xi x) = 0$ and $(\eta x) = 0$ cut (1) in the two binary cubics

$$(2) \quad (a\xi)t^3 + 3(b\xi)t^2 + 3(c\xi)t + (d\xi) = 0,$$

$$(3) \quad (a\eta)t^3 + 3(b\eta)t^2 + 3(c\eta)t + (d\eta) = 0,$$

$$[(a\xi) = a_0\xi_0 + a_1\xi_1 + a_2\xi_2].$$

The combinants of (2) and (3) are expressible in terms of the two-row determinants of the matrix

$$(4) \quad \begin{vmatrix} (a\xi) & (b\xi) & (c\xi) & (d\xi) \\ (a\eta) & (b\eta) & (c\eta) & (d\eta) \end{vmatrix},$$

and if the coordinates of the point in which (ξx) and (ηx) intersect are replaced by x_0, x_1, x_2 respectively, then any two-row determinant of (4) such as $\begin{vmatrix} a_{\xi}^{\eta} & b_{\xi}^{\eta} \\ a_{\eta}^{\xi} & b_{\eta}^{\xi} \end{vmatrix}$ becomes the three-row determinant

$$(5) \quad |abx| = \begin{vmatrix} a_0 & b_0 & x_0 \\ a_1 & b_1 & x_1 \\ a_2 & b_2 & x_2 \end{vmatrix}.$$

Consequently, corresponding to the combinant of (2) and (3) are curves which are covariants of the R^3 .

The Jacobian of (2) and (3) is a binary quartic in t whose coefficients are of degree one in three-row determinants of the type (5). Let us call it $(\alpha t)^4 = 0$. Its roots are the parameters of the four tangents from any point to the R^3 .

If we form the osculant conic of (1) at a point t' , its point equation contains t' to the 4th power. If now t' is made variable, we have a second binary quartic in t whose coefficients are of degree two in determinants of type (5). Let us call it $(\beta t)^4 = 0$; for any given point this gives the parameters of the four osculant conics through the given point.

If P is the covariant line of R^3 cutting out its flexes, and Q the product of the three flex tangents of the R^3 , then the simultaneous invariants of $(\alpha t)^4 = 0$ and $(\beta t)^4 = 0$, which are combinants of (2) and (3), are expressible in terms of P and Q . In this way covariant curves of the R^3 defined by projective relations among the roots of $(\alpha t)^4 = 0$ and $(\beta t)^4 = 0$ are expressed in terms of P and Q and their geometrical interpretation is given.

22. It is the purpose of Professor Mitchell's paper to discuss some of the properties of the finite projective plane $PG(2, 2^n)$ and in particular to determine all subgroups of the group of all projective collineations in $PG(2, 2^n)$. If an outside point of a conic be defined as any point of intersection of tangents to the conic, it is found that in $PG(2, 2^n)$ every conic has but one outside point and all tangents to the conic concur at that point. The types of collineations in $PG(2, 2^n)$ are found to be the same as in the ordinary projective plane, except that in Type I all of the fixed elements may be imaginary. It is shown that every subgroup of the group G_{60480} of all projective collineations in the $PG(2, 2^n)$ except a self-conjugate G_{23160} leaves in-

variant a real figure or an imaginary triangle. Subgroups are determined which leave invariant a point, a line, and an imaginary triangle. The other subgroups are the simple G_{360} , the Hessian G_{216} , the simple G_{168} , and their subgroups. The G_{360} , G_{216} , and G_{168} are all consequently subgroups of the G_{60480} and in the $PG(2, 2^7)$ the geometric invariant of each group is a real configuration.

23. In volume 11, number 4 of the *Transactions*, Professor Westlund published an article on the fundamental number of the field $k(\sqrt[m]{m})$ in which he showed the nature of the decomposition of p and the rational prime factors of m in this field. In the present paper Dr. Wahlin shows how any rational prime, different from these, is decomposed into ideal prime factors in the same field, giving the number of factors and their degrees.

24. The Plimpton library contains several algebraic manuscripts of the fifteenth century. Doubtless the most important of these is the Italian manuscript on vellum which contains numerous biographical notes of Italian teachers of mathematics prominent in the fourteenth and fifteenth centuries. The study made by Professor Karpinski of this Algebra shows that the first chapter dealing with algebra is largely a translation of Mohammed ben Musa Al-Khowarizmi's Algebra. This may well be the translation supposed to have been made by William de Lunis although the reference here to this man is not sufficiently definite to settle the matter. The fourteenth chapter (second on algebra) presents 114 problems from a work on algebra made by an otherwise unknown Maestro Biagio. The fifteenth chapter is from the Algebra of Leonard of Pisa.

This paper will appear in the *Bibliotheca Mathematica*.

25. The name hyperconical connex in space of r dimensions is used in Professor Sisam's paper to denote the locus of the pairs of points (x) and (y) in space of r dimensions whose coordinates satisfy an equation $F(x_0, \dots, x_r; y_0, \dots, y_r) = 0$, where F is homogeneous of degree m in $x_0, \dots, x_r$ and of degree n in $y_0, \dots, y_r$ and satisfies the condition that, if the point (x) is fixed, the resulting hypersurface $F(y) = 0$ is a hypercone with vertex at (x) . It is shown that if $m > n$, $F = 0$ determines the principal coincidence of a point-line connex, and if $m = n$ it determines a line complex. These loci are then studied by means of the above equation.

27. The first paper of Professor Van Vleck gave for a system of linear difference equations a generalization of the following well-known theorem of Poincaré: If the coefficients $P_i(n)$ of the difference equation

$$u(n+m) + P_1(n)u(n+m-1) + \dots + P_m(n)u(n) = 0$$

have for $n = +\infty$ the limits A_i , and if the roots of the corresponding auxiliary equation

$$x^m + A_1x^{m-1} + \dots + A_m = 0$$

are no two of equal moduli, then the ratio $u(n)/u(n-1)$ will have a limit for $n = +\infty$, and this limit will be one of the roots of the auxiliary equation.

28. Professor Van Vleck's second paper sought a basis for the classification of collineations in the iteration of the given finite collineation. This basis has the advantage of bringing into close connection the classifications from the standpoints of Segre and of Lie, and is worked through in detail for space of three dimensions. It is expected that the paper will appear in the near future, and a lengthy abstract is therefore omitted.

29. Mr. Schweitzer investigates the philosophy which underlies the introduction of Grassmann's *Ausdehnungslehre* of 1844. Its most striking feature is perhaps its manifoldness: conceptions of Parmenides, Heracleitos, Plato, Aristotle, Descartes, Kant, Jacobi (cf. Fries) are clearly evident. The key to this variety is probably found in the mediating philosophy of Schleiermacher, who seems to have had fundamentally a Heraclitean motive. Without attempting to express in final or most reduced form the set of general conceptions of Grassmann's philosophy, the author roughly divided these into two classes, objects and acts.

I. Objects: opposition and harmony, being and becoming, representation (copy), flow, change, uniformity, law, valuation, purpose, predominance. Subsidiary to opposition are: within and without, preceding and following, general and particular, one and many, identity and diversity.

II. Acts: thought, division, analysis, generation, synthesis, giving and coupling.

In his well-known treatise on geometry Veronese seems closely to follow Grassmann's philosophical introduction as a general basis for his specific development.

30. In his second paper Mr. Schweitzer considers methods of control of mathematical content. He discussed the necessity of universals for the (genetic) logic of mathematics and referred to Grassmann's "guiding conception," the "intuition" of Klein, and Moore's "heuristic principle" as expressions of this need. Finally, Russell's Principles of Mathematics was examined for evidences of induction.

31. Let $f(x)$ be a function of the complex variable x analytic within and upon the boundary of a certain infinite region T , exception being made of the point $x = \infty$. Under these circumstances, $f(x)$ may or may not be developable asymptotically throughout T in the form

$$(1) \quad f(x) \sim g_0 + g_1 x^{-1} + g_2 x^{-2} + \dots$$

It is the object of Professor Ford's paper to determine a set of sufficient conditions for $f(x)$ under which the existence of the development (1) will be assured. Two theorems are established, corresponding to the cases in which x is complex or confined to a real interval, and application is made to certain studies connected with the existence of asymptotic solutions for linear differential equations.

32. It can be shown that with respect to the irreducible surfaces of lowest order which may contain the curve as partial intersection, irreducible twisted quintic curves can be classified as follows:

$$(1) \ C_6 + R_1 = (S_2, S_2), \quad (2) \ C_6 + R_3 = (S_2, S_2),$$

$$(3) \ C_6 + R_4 = (S_2, S_2),$$

a) R_2 , quartic of the second kind,

b) R_2 , improper quartic of the second kind,

where C_6 denotes the curve in question; R_λ the residual intersection of order λ , and S_μ, S_ν the surfaces of order μ and ν whose complete intersection is $C_6 + R_\lambda$.

Mr. Montgomery effects the classification of all irreducible twisted quintics by means of the system $(m, n, r, h, H, g, G, x, y, \alpha, \beta, \gamma, L)$, which are characteristics of the system of points, lines, and planes of the developable of which the twisted curve is the cuspidal edge. These characteristics are connected by the six Plücker-Cayley equations which, involving 13

quantities, are not sufficient for the complete solution ; and consequently two of the characteristics, γ and L , are independently considered. The curves are also classified with respect to monoids, and a special discussion of the quintic with a triple point is given. There are thus obtained 42 different irreducible twisted quintic curves.

33. The object of Professor Marshall's paper is to present a method of solving Hill's differential equation for large values of the argument. A special case is considered in detail, the form of the function determined, and the recurrence formulas for the coefficients given. Incidentally it appears that the results obtained include as special cases the asymptotic expansions for Bessel's functions, for the elliptic cylinder functions, and for the harmonic functions associated with the parabolic cylinder. The application of the method to the general equation is briefly discussed.

34. It was Lie's belief that the processes which take place in nature can be represented mathematically by a succession of infinitesimal transformations for which the laws of the universe are the invariants. In Mr. Bateman's paper, Lie's idea is applied to an electromagnetic field and transformations which convert the field into itself are obtained on the supposition that three particular integral forms remain invariant in the passage from one view of the universe to another.

35. In his dissertation (Chicago, 1907) Dr. Lennes defined a continuous arc connecting two points A and B as a bounded, closed, connected set of points containing A and B but having no connected proper subset containing A and B . In the present paper it is shown that this definition applies without any change whatever to arcs in space. A simply connected closed continuous surface is defined as a closed, bounded, finitely connected set $[S]$ of points such that

- (a) No simply connected subset separates the remaining points.
- (b) Every doubly connected subset separates the remaining points into two connected sets.

It is proved that this set of points may be mapped continuously on a tetrahedron and that it separates a three-space in which it lies into two connected sets. It is further shown that obvious modifications of the above definition give a multiply connected surface.

36. Ascoli has shown that if an infinite set of curves connect two fixed points, a necessary and sufficient condition that every infinite sequence of curves of this set shall approach at least one limiting curve uniformly is that the set of curves is *equally* continuous. In Dr. Lennes's second paper this theorem is extended to surfaces and also to curves whose end points are in two separate closed sets of points and not fixed as in Ascoli's theorem.

The generalized theorem is then used to make proofs of very general theorems on the existence of minimizing curves and surfaces. The existence of the limiting curves and surfaces is treated from the non-metric point of view.

37. Dr. Neikirk's paper is a continuation of that presented by him to the Chicago Section at Minneapolis in December. It contains a more detailed study of the properties of infinite substitutions and the generalization of several important theorems on rational functions.

38. From a point (ξ, η) in the plane n tangents proceed to a parabolic curve of the form $y = a_0 + a_1x + \dots + a_nx^n$. Let k represent the projection on the x axis of the tangent length from the point of contact to (ξ, η) . Professor Maclay's paper determines the elementary symmetric functions of the k 's in terms of the point (ξ, η) and the coefficients of the curve. The relations of the symmetric functions and functions of the symmetric functions to movements of the point (ξ, η) and to transformations of the plane are treated.

Two results may be stated. For any point (ξ, η) and for every parabolic curve $\Sigma(1/k) = 0$. The second result is expressed in the equation $k_1k_2 \dots k_n = L/(n-1)a_n$ in which L is the distance from (ξ, η) to the curve measured on a parallel to the y axis. For the ordinary parabola, i. e., $n = 2$, the elementary symmetric functions reduce to the two just given. Interpreted with reference to the parabola, they state that if the parabola be translated parallel to the direction of its principal diameter, the intercepted tangent lengths between the old and new positions project into equal lengths on a line perpendicular to the principal diameter.

39. In his Projective Differential Geometry of Curves and Ruled Surfaces Wilczynski has given a geometrical interpretation

to the seminvariants, semi-covariants, invariants, and covariants of a linear differential equation of the third order. The fundamental solutions y_1, y_2, y_3 of the differential equation are interpreted as homogeneous coordinates of a point P_y in a plane; as the independent variable changes, P_y describes a curve C_y . The semi-covariants, z and ρ , define points P_z and P_ρ in the plane. Taking P_y, P_z, P_ρ as the vertices of a triangle of reference, the equation of the osculating conic at P_y of the curve C_y assumes a very simple form. The application of these interpretations, the determination of the cartesian equation of the osculating conic, of the Halphen point of P_y , and the locus of a covariant expression P_z , when C_y is a cubical parabola, semi-cubical parabola, cissoid, and folium, are the subject of Mr. Nyberg's paper.

40. Professor Wilczynski has proved that the theory of non-developable ruled surfaces in ordinary space can be put into relation with the theory of a system of two linear homogeneous differential equations of the second order. Mr. Stouffer extends this notion to ruled surfaces in five-dimensional space and shows that the projective properties of every surface of this kind, which does not belong to a certain exceptional class, can be obtained by studying a system of two linear homogeneous differential equations of the third order. The ruled surfaces to which the theory does not apply are those which have three consecutive generators in the same space of four dimensions.

41. Given a system of differential equations

$$(A) \quad \frac{dx_i}{dt} = f_i(x_1, \dots, x_n; \mu; t) \quad (i = 1, \dots, n)$$

in which the f_i are analytic in x , and μ and regular for $x_i = \mu = 0$, and, with respect to t , are continuous and periodic with the period 2π ; the Cauchy-Poincaré existence theorem states that the solutions of (A) exist and can be expanded as power series in μ and the initial values of $x_1, \dots, x_n$, the coefficients of these power series being functions of t .

Dr. MacMillan shows that if

$$(B) \quad \frac{dx_i}{dt} = \sum_{j=1}^n \theta_{ij} x_j,$$

are the linear terms of (A), then the solutions of (A) are ex-

pansible as power series in μ and the constants of integration of the system (B) and converge if these constants are sufficiently small.

By means of the expansions in this form it is shown that if periodic solutions with the period $2\kappa\pi$ as power series in the parameter μ

$$(C) \quad x_i = \phi_i(\mu; t)$$

can be found which formally satisfy (A), then the solutions (C) exist and converge for all values of t , provided μ is sufficiently small.

42. The problem of the position of equilibrium of a heavy cord with one end point fixed and the other movable on a given curve has been discussed by Kneser as a problem in the calculus of variations. Dr. Crathorne considered this problem for both end points movable on given curves K_1 and K_2 , and in particular discussed the Jacobi condition. The resulting curve is a catenary intersecting K_1 and K_2 at right angles. The relations between the critical points and the curvature of K_1 and K_2 at their intersections P_1 and P_2 with the catenary were given. In the case in which the catenary approaches the convex side of both the fixed curves the Jacobi condition is satisfied. In the other cases the fulfillment of this condition depends on the curvatures of K_1 and K_2 at P_1 and P_2 . Also, if P'_1 and P'_2 represent the critical points of K_1 and K_2 respectively, then P'_1 cannot lie between P_1 and P'_2 . This is analogous to Bliss's condition in the non-isoperimetric problem.

43. Periodic solutions of the problem of three bodies fall into a large number of classes which require somewhat different modes of treatment. The class of orbits considered by Professor Moulton in this paper is that in which two finite bodies describe circles, while a distant infinitesimal body moves in the plane of their motion subject to their attraction. The orbits discussed are those whose period is the synodic period of revolution of the infinitesimal body. The solutions are regular analytic functions of various parameters among which is the energy constant. For a given value of the constant there are six geometrically distinct orbits in three of which the motion is direct and in three of which it is retrograde. For small values of the parameter only two of the orbits are real.

44. The differential equation treated in Professor Moulton's second paper is

$$\frac{d^2x}{dt^2} = \sum_{i=-\infty}^{+\infty} a_i x^i,$$

where the a_i are constants. This equation includes as special cases those defining elliptic and hyperelliptic functions, as well as many others which arise in physical problems. Discussion is made particularly of the solutions which are periodic. Convenient methods of obtaining their periods and explicit representations are developed. In case the right member is a polynomial in x the precise domain of validity is found. These results include new and useful expansions of the elliptic functions. The Legendre elliptic functions of real period are explicitly developed to the twelfth power of the modulus inclusive, and it is proved they are valid for all real finite values of t .

45. The transformation employed in Professor Safford's paper is quadric and irreducible. It is based upon a formula given in Weierstrass's lectures and published in a dissertation by G. G. A. Biermann. This formula may also be found in Enneper, *Elliptische Functionen*, § 5, and in Greenhill, *Elliptic Functions*, § 150.

46. Professor Roever derived for the southerly deviation of a falling body the expression

$$(1) \quad \frac{4 \sin \phi \cos \phi \omega^2 - 5(\partial g / \partial \xi)_0 h^2}{6g_0},$$

in which h is the height through which the body falls, ϕ and g_0 the latitude and the acceleration respectively at the point P_0 from which the body falls, ω the angular velocity of the earth's rotation, and ξ (southerly) distance measured along the meridian which passes through P_0 . If $f_1(r, z)$ (where r represents the distance of a general point from the earth's axis of rotation and z that from a plane π perpendicular to this axis) represents the potential function of the field of force F_1 , under the influence of which the falling body moves after its connection with the rotating earth has been severed,*

* The potential function of the field of force F_2 , under the influence of which the plumb-line is in equilibrium, is $f_2(r, z) = f_1(r, z) + \frac{1}{2}\omega^2 r^2$.

$$g_0 = \sqrt{\left(\frac{\partial f}{\partial z}\right)_0^2 + \left[\left(\frac{\partial f}{\partial r}\right)_0 + \omega^2 r_0\right]^2}, \quad g_0 \sin \phi = \left(\frac{\partial f}{\partial z}\right)_0,$$

$$g_0 \cos \phi = \left(\frac{\partial f}{\partial r}\right)_0 + \omega^2 r_0,$$

$$\left(\frac{\partial g}{\partial \xi}\right)_0 = \sin \phi \cos \phi \left[\left(\frac{\partial^2 f}{\partial z^2}\right)_0 - \left(\frac{\partial^2 f}{\partial r^2}\right)_0 \right]$$

$$+ (\cos^2 \phi - \sin^2 \phi) \left(\frac{\partial^2 f}{\partial r \partial z}\right)_0 - \sin \phi \cos \phi \omega^2,$$

the subscript $_0$ denoting that the derivatives are computed for the point $P_0(r=r_0, z=z_0)$. Expression (1) is equal to the difference $\xi_1 - \xi_2$, where

$$\xi_1 = \frac{4 \sin \phi \cos \phi \omega^2 + (\partial g / \partial \xi)_0}{6g_0} h^2, \quad \xi_2 = \frac{(\partial g / \partial \xi)_0}{g_0} h^2.$$

In order to show what ξ_1 and ξ_2 represent, let us denote by R the plumb-bob of the plumb-line which is supported at P_0 and has the length $h = P_0 R$, and by T and S the points in which the direction of gravity at P_0 and the path of the falling body pierce the level surface of R , respectively. Then ξ_1 and ξ_2 are the distances, measured along a meridian, which the circles of latitude of S and R lie to the south of the circle of latitude of T , and the southerly deviation of the falling body is the distance which the circle of latitude of S lies to the south of that of R . In existing formulas the part ξ_2 has been entirely neglected. In the formula of Gauss ($\frac{2}{3} \sin \phi \cos \phi \omega^2 / g_0 \cdot h^2$) this is due to the assumption that g is constant and therefore that $(\partial g / \partial \xi)_0 = 0$. For the potential function for which the standard spheroid is a level surface,† $-\xi_2$ is eighteen times as great as ξ_1 , and the southerly deviation $\xi_1 - \xi_2$ is four and one half times as great as the expression given by Gauss.

47. Cesàro's definition of a series k times indeterminate for the case of a simple series can be extended to double series.

$$\dagger f_1(r, z) = \frac{M}{\rho} \left[1 + \frac{\varepsilon \rho_1^2}{3 \rho^2} \frac{r^2 - 2z^2}{\rho^2} \right],$$

where $\rho^2 = r^2 + z^2$, $\varepsilon = .00167$, M and ρ_1 are mass and mean radius of earth. See Poincaré, *Figures d'Equilibre d'une Masse fluide* (1902), Chap. V.

We form

$$S_{m,n}^{(k)} = \sum_{i=0}^{m-n} \sum_{j=0}^{n-i} \binom{k+m-i-1}{k-1} \binom{k+n-j-1}{k-1} s_{i,j},$$

where

$$s_{m,n} = \sum_{i=0}^m \sum_{j=0}^n a_{i,j},$$

and

$$A_{m,n}^{(k)} = \binom{m+k}{k} \binom{n+k}{k}.$$

Then if $S_{m,n}^{(k)} / A_{m,n}^{(k)}$ approaches a limit as m and n become infinite, whereas $S_{m,n}^{(k-1)} / A_{m,n}^{(k-1)}$ does not, we say that the series $\sum a_{i,j}$ is k times indeterminate, and that its sum is equal to the above limit.

The principal result of Professor Moore's paper is the proof of the following theorem: If the series $\sum a_{i,j}$ is k times indeterminate with sum S and furthermore $S_{m,n}^{(k)} / A_{m,n}^{(k)}$ remains finite for all values of m and n , then the series $\sum a_{i,j} f_{i,j}(\alpha, \beta)$ will converge for positive values of α and β , its value $F(\alpha, \beta)$ will be continuous for such values, and finally

$$\lim_{\alpha \rightarrow +0, \beta \rightarrow +0} F(\alpha, \beta) = S,$$

provided the convergence factors $f_{i,j}(\alpha, \beta)$ are continuous for $\alpha, \beta > 0$, and satisfy the following conditions:

- (a) $\sum_{\mu_1, \nu_1}^{\mu_2, \nu_2} i^k j^k |\Delta_2^{k+1} f_{i,j}| < K,$
 (b) $\lim_{m \rightarrow \infty} m^k f_{m,n} = 0, \quad \lim_{n \rightarrow \infty} n^k f_{m,n} = 0, \quad \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} m^k n^k f_{m,n} = 0, \quad \alpha, \beta > 0,$
 (c) $\lim_{\alpha \rightarrow +0, \beta \rightarrow +0} f_{i,j}(\alpha, \beta) = 1 \quad (i = 1, 2, 3, \dots; j = 1, 2, 3, \dots),$

where K is a positive constant, $\mu_1, \nu_1, \mu_2, \nu_2$ are any four positive integers, and $\Delta_2^{k+1} f_{i,j}$ is used as an abbreviation for

$$\sum_{\mu=i}^{\mu=i+k+1} \sum_{\nu=j}^{\nu=j+k+1} (-1)^{\mu-i} (-1)^{\nu-j} \binom{k+1}{\mu-i} \binom{k+1}{\nu-j} f_{\mu,\nu}.$$

49. The second paper by Professor Dickson deals with Fermat's sketch of his proof that no right triangle with rational sides can equal a square with a rational side. Fermat (Oeuvres

3, Paris, 1896, page 271) states that if $\xi^2 = \eta^2 + 2b^2$ it is easily proved that ξ itself is of the form $r^2 + 2s^2$. It is of the nature of an anachronism to complete the proof, as did Legendre, *Théorie des nombres*, Paris, 1808, page 342, by means of the theory of quadratic forms, or, as did Bachmann, *Niedere Zahlentheorie* 2, page 451, by means of the properties of the integral algebraic numbers $x + y\sqrt{-2}$. Very likely Fermat had a proof of the following nature. Set $e = \frac{1}{2}(\xi + \eta)$, $f = \frac{1}{2}(\xi - \eta)$. Then e, f , and $ef = \frac{1}{4}b^2$ are integers, and e, f are relatively prime. By changing the sign of η , if necessary, we may assume that e is odd. Then $e = r^2$, $f = 2s^2$, $\xi = e + f = r^2 + 2s^2$.

Fermat then considers the right triangle with the legs r^2 and $2s^2$ (and hence of area a square). Its legs may be expressed in the form $2m_1n_1 = 2s^2$ and $m_1^2 - n_1^2 = r^2$, where $m_1 = a_1^2$, $n_1 = b_1^2$, by the earlier argument. Then $(m_1^2 + n_1^2)^2 = r^4 + 4s^4 = a^4$. Hence $a_1^4 + b_1^4 = a^4$. For integral values, $a_1^2 + b_1^2 \leq a_1^4 + b_1^4$, $a < a^2 + b^2$. Hence we have proved Fermat's assertion that $a_1^2 + b_1^2 < a^2 + b^2$.

51. The first part of Professor Glenn's paper is devoted to the development of a binary form α_x^n ($n = \mu\nu$) in a series of ascending powers of a given form α_x^μ . The general development is obtained. The necessary and sufficient conditions are then established that α_x^n be a sum of multiples of prescribed powers of α_x^μ . A special case of these results is Hilbert's covariant which expresses by its identical vanishing the necessary and sufficient conditions that α_x^n be the ν th power of α_x^μ (*Mathematische Annalen*, volume 27).

52. Professor Clements considers the transformation

$$T: \quad x = f(u, v) \quad y = \phi(u, v),$$

f and ϕ denoting functions of the complex variables u and v , single valued and analytic throughout some neighborhood R of $u = 0, v = 0$, and satisfying the conditions

$$f(0, 0) = 0, \quad \phi(0, 0) = 0,$$

$$J(0, 0) = (f_u \phi_v - f_v \phi_u)_{u=0, v=0} \neq 0 \quad [\neq 0]_{u, v}.$$

The behavior of the inverse solution $u = \bar{f}(x, y)$, $v = \bar{\phi}(x, y)$ in the region $\bar{R}$ for (x, y) into which T carries R , is considered.

It is shown that no inverse can exist, single valued and analytic throughout $\bar{R}$; that if f and ϕ admit no common factor in R , there always exists an inverse, continuous and multiple valued throughout $\bar{R}$, the branches coming together with value zero in $x = 0, y = 0$.

A more general set of equations

$$f_i(x_1, \dots, x_n; y_1, \dots, y_p) = 0 \quad (i = 1, 2, \dots, p),$$

where

(1) the f_i are analytic functions of all their arguments in a certain region, and $f_i((0); (0)) = 0$;

$$(2) \quad \frac{D(f_2, \dots, f_p)}{D(y_2, \dots, y_p)} \neq 0 \quad \text{when } (x) = 0, (y) = 0;$$

$$(3) \quad J_1[(x); (y)] \equiv \frac{D(f_1, f_2, \dots, f_p)}{D(y_1, \dots, y_p)} = 0 \quad \text{when } (x) = 0, (y) = 0,$$

$$J_{i-1}[(x); (y)] \equiv \frac{D(J_{i-2} f_2, \dots, f_p)}{D(y_1, \dots, y_p)} = 0 \quad \text{when } (x) = 0, (y) = 0,$$

$$J_k[(x); (y)] \equiv \frac{D(J_{i-1} f_2, \dots, f_p)}{D(y_1, \dots, y_p)} \neq 0 \quad \text{when } (x) = 0, (y) = 0,$$

defines p functions

$$y_1 = \phi_1(x_1, \dots, x_n), \dots, y_p = \phi_p(x_1, \dots, x_n)$$

of $x_1, \dots, x_n$, continuous but not analytic throughout some complete neighborhood of $x_1 = 0, \dots, x_n = 0$. This inverse is multiple valued and has in general k branches. This theorem is applied to the transformation

$$x_i = f_i(y_1, \dots, y_p) \quad (i = 1, 2, \dots, p).$$

Condition (3) requires that the f_i shall have no common factor in the neighborhood of $(y) = 0$. In this case it is shown that the k branches are distinct.

The factorization of transformations is taken up and certain useful types are discussed in detail. It is shown that if T satisfies the further condition $J_1(0, 0) \neq 0$, it can be factored into transformations one-to-one and analytic both ways, plus one of the type $x = u, y = v^2$.

The cases in which the Jacobian determinant of T has at least one non-zero element, say $f_u(0, 0) \neq 0$, are completely discussed. Certain cases where all f_u, f_v, ϕ_u, ϕ_v are zero when $u = 0, v = 0$ are treated. If f and ϕ admit a common factor in R , then there is an explosive point in $\bar{R}$, having an infinitely many valued inverse. Even then $\bar{R}$ may be the complete neighborhood of this point, the number of branches which are continuous outside this point being different in different sub-regions of $\bar{R}$.

53. It is well known that the group of isomorphisms of a group of order p is of order $p - 1$, and that of a cyclic group of order p^2 is of order $p(p - 1)$. The corresponding group of the non-cyclic group of order p^2 is simply isomorphic with the linear homogeneous group on p^2 variables.

The groups of isomorphisms of all types of groups of order p^3 are determined by Western in his paper on "Groups of order p^3q ," *Proceedings of the London Mathematical Society*, volume 30.

Professor Marriott has determined the groups of isomorphisms of all types of groups of order p^4 . He exhibits these as substitution groups and determines the order of each.

F. N. COLE,
Secretary.

ON THE NEGATIVE DISCRIMINANTS FOR WHICH THERE IS A SINGLE CLASS OF POSITIVE PRIMITIVE BINARY QUADRATIC FORMS.

BY PROFESSOR L. E. DICKSON.

(Read before the American Mathematical Society, April 29, 1911.)

FOR such a discriminant $-P$, the problem of the representation of numbers by a binary quadratic form of discriminant $-P$ is quite elementary; moreover, factorization into primes is unique in a quadratic field of discriminant $-P$. The only*

*E. Landau, *Mathematische Annalen*, vol. 56 (1903), p. 671. His method is not applicable to discriminants $-P$, where P is odd, as was pointed out by M. Lerch, *ibid*, vol. 57 (1903), p. 568. Results obtained by the latter by use of a relation between numbers of classes will here be proved by more elementary means and extensions given.

such discriminants of the form $-4k$ are those having $k = 1, 2, 3, 4, 7$, as was conjectured by Gauss* after an examination of the *determinants* as far as -3000 . The present note gives practical criteria and the result of an examination of the values of P less than one and one half million. We denote $ax^2 + bxy + cy^2$ by (a, b, c) and call $b^2 - 4ac$ its discriminant.

First, let $P \equiv 0 \pmod{4}$. Then $(1, 0, P/4)$ must be the only reduced primitive form of discriminant $-P$. The case in which $P/4$ is divisible by two distinct primes is excluded, since we may then express $P/4$ as the product of two relatively prime factors a, c , such that $1 < a < c$, and hence obtain the new primitive reduced form $(a, 0, c)$ of discriminant $-P$. Hence $P = 4p^e$, where p is a prime. For $p = 2, (4, 4, 2^{e-2} + 1)$ is a primitive reduced form of discriminant $-P$ if $e \geq 4$, and $(3, 2, 3)$ is one if $e = 3$; while for $e = 1$ or 2 , whence $P = 8$ or 16 , there is a single primitive reduced form. Next, let $p > 2$. The even number $p^e + 1$ cannot have an odd factor > 1 , since otherwise it would equal the product of two relatively prime integers a and c , such that $1 < a < c$, and $(a, 2, c)$ would give a new primitive reduced form of discriminant $-P$. Hence $p^e + 1 = 2^k$. Then $(8, 6, 2^{k-3} + 1)$ or $(5, 4, 7)$ is a primitive reduced form of discriminant $-P$ if $k > 5$ or $k = 5$, respectively. For $k = 4, 2^k - 1 = 15$ is not a power of a prime. For $k = 1, 2, 3, P = 4, 12, 28$, there is a single primitive reduced form.

Next, let $P \equiv 3 \pmod{4}$. Then $[1, 1, \frac{1}{4}(1 + P)]$ must be the only reduced primitive form of discriminant $-P$. If $P = rs$, where r and s are relatively prime and > 1 , one of the factors is $\equiv 3 \pmod{4}$ and the other $\equiv 1 \pmod{4}$. Let $r > s$. Then $[(r + s)/4, (r - s)/2, (r + s)/4]$ is a new primitive form of discriminant $-P$, which is reduced if $3s \geq r$. Its second right neighboring form (obtained by using $\delta = -1, \delta' = 0$) is $[s, -s, \frac{1}{4}(r + s)]$, which is reduced if $3s < r$. Hence $P = p^e$, where p is a prime $\equiv 3 \pmod{4}$ and e is odd. If $p > 3, e \geq 3$, the form with $a = \frac{1}{4}(p + 1), b = 1, c = (p^e + 1)/(p + 1)$ is a new primitive reduced form of discriminant $-P$; indeed, $c > 4a$ since $p^{e-1} \geq p^2 > p + 2$. For $P = 27, (1, 1, 7)$ is the only primitive reduced form. For $P = 3^e, [9, 3, \frac{1}{4}(3^{e-2} + 1)]$ or $(7, 3, 9)$ is a primitive reduced form if $e > 5$ or $e = 5$, respectively. Thus, if $P \neq 27, P$ must be a prime. Set

$$T_j = \frac{1}{4}[(2j + 1)^2 + P] \doteq T_0 + j(j + 1).$$

* Disquisitiones Arithmeticae, Art. 303.

If $j = gm + r$, $0 \leq r < m$, then $T_j \equiv T_r \equiv T_{m-r-1} \pmod{m}$. For $r > \frac{1}{2}(m-1)$, $m-r-1 < \frac{1}{2}(m-1)$. Hence any T_j is congruent modulo m to some T_r , where $0 \leq r \leq \frac{1}{2}(m-1)$. Let $2g+1$ be the greatest odd integer $\leq \sqrt{P/3}$. In a reduced form (a, b, c) , $b > 0$, we have $b = 2\beta + 1 \leq 2g + 1$, $\beta \leq g$. We shall prove that *there is a single reduced form of discriminant $-P$ if and only if $T_0, T_1, \dots, T_g$ are all prime numbers*. If they are primes, a reduced form has $a=1, b=1$. Conversely, let there be a single reduced form. If T_0 were composite, there would be a reduced form with $b=1, a>1$. Suppose that $T_0, \dots, T_{\beta-1}$ are primes, but $T_\beta = ac$, $c \geq a > 1$, where $0 < \beta \leq g$. If $a \geq b$, where $b = 2\beta + 1$, (a, b, c) would be reduced. Hence $a < b$. Applying the above result for $m=a$, we see that $T_\beta \equiv T_r \pmod{a}$, where r is some integer $0 \leq r \leq \frac{1}{2}(a-1)$. Thus $r < \beta$, so that T_r is a prime. But $T_r \equiv T_\beta \equiv 0 \pmod{a}$. Hence $T_r = a$. Thus $a \geq T_0 \geq \frac{1}{2}(1+P)$. $P \geq 3(2g+1)^2 \geq 3(2\beta+1)^2 > 3a^2$, $a > \frac{1}{2}(1+3a^2)$. Thus $(3a-1)(a-1) < 0$, which contradicts $a > 1$.

If P is a prime < 27 , then $g=0$ and the condition is that $T_0 = \frac{1}{2}(1+P)$ shall be a prime. This is satisfied when $P = 3, 7, 11, 19$.

For $P \equiv 7 \pmod{8}$, $P > 7$, T_0 is even and > 2 .

For $P \equiv 3 \pmod{8}$, set $P = 8k - 5$. For $k \equiv 2 \pmod{3}$, $k \geq 5$, $T_0 = 2k - 1$ is divisible by 3 and exceeds 3; while for $k=2$, $P=11$. For $k \equiv 1 \pmod{3}$, P is divisible by 3. For $k \equiv 0 \pmod{3}$, $P = 24t - 5$. For $t \equiv 1, 4, \text{ or } 0 \pmod{5}$, $T_0 = 6t - 1$, $T_1 = 6t + 1$ or P is divisible by 5 and exceeds 5 except when $t=1$, $P=19$. For $t=2$ or 3 , $P=43$ or 67 and $g=1$, while T_0 and T_1 are primes. For $t=7$, $P=163$, $g=3$, and T_0, T_1, T_2, T_3 are primes 41, 43, 47, 53. For $t=8$, $P=11 \cdot 17$. There remain the cases $t=5l+12, 5l+13$, where $l \geq 0$. Hence we may state the

THEOREM. *There is a single class of positive primitive quadratic forms of negative discriminant $-P$ when*

$$P = 3, 4, 7, 8, 11, 12, 16, 19, 27, 28, 43, 67, 163;$$

but more than one class if P is not one of these 13 numbers and not a prime of the form $120l + 283$ or $120l + 307$, $l \geq 0$.

The remaining primes < 1000 are $P = 283, 523, 643, 883, 307, 547, 787, 907$. For these $g \geq 4$, $T_2 = 77$, $T_1 = 7 \cdot 19$, $T_0 = 7 \cdot 23$, $T_0 = 13 \cdot 17$, $T_0 = 77$, $T_2 = 11 \cdot 13$, $T_3 = 7 \cdot 29$,

$T_4 = 13 \cdot 19$, respectively. Hence in each case there is more than one class.

A practical method of examining a wide range of values of P consists in first excluding the values of l for which any one of the numbers $T_0, \dots, T_g$ has a given small prime factor p . For $P = 120l + 283$ or 307 , $T_0 = 30l + 71$ or $30l + 77$, $g \geq 4$. This exclusion has already been effected for $p = 3$ or 5 . For $p = 7$, any T_j is congruent to T_0, T_1, T_2 or T_3 . For $T_0 = 30l + 77$, these are divisible by 7 if $l \equiv 0, 6, 4, 1 \pmod{7}$, respectively; for $30l + 71$, if $l \equiv 3, 2, 0, 4 \pmod{7}$. Hence there remain the cases

$$T_0 = 210m + \mu, \mu = 137, 167, 227, 101, 221, 251.$$

The least P is now 403, whence $d \geq 5$. Now $T_0 \equiv m + \mu \pmod{11}$. Thus T_0 is divisible by 11 if $m \equiv 6, 9, 4, 9, 10, 2 \pmod{11}$, respectively. But $T_k = T_{k-1} + 2k$. Hence if we subtract $2k$ from the m for which $T_{k-1} \equiv 0 \pmod{11}$, we obtain the m for which $T_k \equiv 0 \pmod{11}$. This may be done by counting spaces on square ruled paper. At each point so obtained a hole is punched, thus giving a 6×11 stencil for $p = 11$. The least T_0 is now 221, whence $P \geq 883$, $g \geq 8$. Similarly, stencils were constructed for $p = 13, 17, 19, 23, 29$. After using the first three stencils, it was noted that $m \geq 4$ for each μ , whence $T_0 \geq 941$, $P \geq 3763$, $g \geq 17$.

The first 10710 values of T_0 were examined; to this end m was given the values ≤ 1785 . The use of each stencil excluded more than half of the values left at the earlier stage. After using the stencils for $p \leq 29$, we had left 110 numbers, for each of which $T_0, \dots, T_g$ was verified to be composite. In just four cases were $T_0, \dots, T_g$ all prime. The work, including the making of the stencils, was done in two days.

THEOREM. *For $163 < P < 1,500,000$ there is more than one class of positive primitive quadratic forms of discriminant $-P$.*

For a greater P , $g \geq 353$ and there is more than one class unless $T_0, T_1, \dots, T_{353}$ are all primes. The chance that such a case will arise is extremely small. Note that, for P not exceeding $1\frac{1}{2}$ millions, $T_0, \dots, T_{14}$ were shown to be not all prime.

ITERATED LIMITS OF FUNCTIONS ON AN ABSTRACT RANGE.

BY MR. RALPH E. ROOT.

(Read before the American Mathematical Society, April 29, 1911.)

RESERVING details for a later publication, I wish here to indicate a method for the investigation of multiple and iterated limits under general conditions imposed on the range of the independent variable, and free from the unnecessary restrictions involved in the consideration of variables of particular types. For desirable simplicity of the general theory the postulates imposed on the range should not only permit of an effective definition of limiting element, but also be invariant under two fundamental processes:

(a) Extension of the system through the adjunction of ideal limiting elements.

(b) Composition of systems, e. g., the generation from two linear ranges of a two dimensional range.

In his thesis (Paris, 1906): "Sur quelques points du calcul fonctionnel" (*Rendiconti del Circolo Matematico di Palermo*, volume 22) M. Fréchet supposes the notion of limit of a sequence of elements to be defined for a certain abstract class and secures generalizations of several theorems on point sets and on continuous functions. Later, adopting an analogue of the distance relation (écart) between two points, he secures many more theorems. His postulates, however, do not readily admit of the introduction of ideal elements.

F. Riesz, in a paper before the International Congress of Mathematicians at Rome, 1908 ("Stetigkeitsbegriff und abstracte Mengenlehre," *Atti*, volume 2 (1908) pages 18-24) proposed a set of postulates in which the fundamental notion is that of a relation between subclass and element of the range, in the sense that the element is a "limiting" element of the subclass. By considering the properties of a system of subclasses that have a common limiting element he is able to define effectively an ideal limiting element as a system of subclasses having certain definite properties. The postulates of Riesz, however, are apparently better fitted for the treatment of abstract point set theory than for the treatment of functions on the range thus characterized.

Let $\mathfrak{P}$ be a class of elements, and let R represent a dyadic relation between subclasses $\mathfrak{R}$ of $\mathfrak{P}$. For the system $(\mathfrak{P}; R)$ we adopt a body of postulates that permit of a very simple definition of limit of a function at an element, in which the classes $\mathfrak{R}$ that have the relation R to a given singular class p (element of $\mathfrak{P}$) play the rôle of "vicinity" or "neighborhood" of a point. We are able to characterize a class u of subclasses $\mathfrak{R}$ as an ideal element in such fashion that the postulates are satisfied by the extended system $(\mathfrak{Q}; S)$, where $\mathfrak{Q}$ is the class $\mathfrak{P}$ with ideal elements adjoined, and S is a relation between subclasses $\mathfrak{S}$ of $\mathfrak{Q}$ defined in terms of R . The extended system $(\mathfrak{Q}; S)$ is closed to this process of extension.

Further, from two such systems, $(\mathfrak{P}'; R')$ and $(\mathfrak{P}''; R'')$, we derive a composite system $(\mathfrak{P}; R)$, where $\mathfrak{P}$ is the product class $\mathfrak{P}'\mathfrak{P}''$ and R is a relation between subclasses of $\mathfrak{P}$ determined by R' and R'' . Our postulates are satisfied by the composite system if, and only if, they are satisfied by both component systems. This process of composition may be generalized to give a unique composite system $(\mathfrak{P}; R)$ for any finite number of systems $(\mathfrak{P}'; R')$.

In terms of our relation R we define limit for a sequence of elements so as to fulfil the conditions of the Fréchet limit, and our definition of limiting element of a subclass fulfils the Riesz postulates. We show that in a system satisfying our postulates the definitions of continuity by means of "sequences" and by means of "vicinity" are equivalent. On the other hand our system is less special than the system of Fréchet in which he uses the notion of *écart*, since a relation R fulfilling our conditions may be defined for any class for which an *écart* exists. The gain in generality over the Fréchet treatment is seen in the possibility of determining the relation R by means of order relations in which distance and magnitude play no part. An R may be defined for any class having linear order, or for the composite of any finite number of such classes.

CHICAGO, ILL.
April, 1911.

NOTE ON A MERSENNE NUMBER.

BY MR. H. J. WOODALL, A.R.C.S.

I DESIRE to announce the discovery that $2^{181} - 1$ is divisible by 43441 (a prime). This is important because the exponent 181 is one of the few remaining prime numbers, less than 257, about which it was to be inferred, as not being included in Mersenne's list, that, used as exponents, they would give $2^p - 1$ composite numbers. The writer has verified the result by direct division, and desires to express his thanks to Lieutenant Colonel Allan Cunningham, R.E., for his verification.

MARKET PLACE,
STOCKPORT, ENGLAND,
June 7, 1911.

SHORTER NOTICES.

A Text-Book on Advanced Algebra and Trigonometry, with Tables. By W. C. BRENKE, Ph.D. New York, The Century Company, 1910. 8vo. x + 345 pp.

PROFESSOR Brenke has succeeded in bringing together within the 284 pages of his book that are taken up by the text most of the topics of algebra and trigonometry that are suitable for presentation in the freshman year of our colleges. Selection of subjects can be made so as to suit the needs of the classes in liberal arts colleges, where only three hours a week are devoted to mathematics, as well as those of classes in the colleges of engineering, where five weekly hours is the allotted time.

As the title indicates, this book is an attempt to correlate the subjects of algebra and trigonometry. The author says in the preface: "Usually algebra is taken up first and then trigonometry, or else the two subjects are studied on alternate days. Neither plan is quite satisfactory. It has therefore seemed to the writer that a single book, treating both subjects in a correlated manner, might be of service both to student and teacher."

In the professional colleges correlation is induced by the fact that the work in mathematics is to be applied later on; the desire to direct the work as much as possible toward the appli-

cations will tend to unify and correlate the work in these colleges, independently to some extent at least of how the subjects are arranged in the text-book. It is in the non-professional colleges, where for many the freshman work in mathematics is the last they will ever do in that subject, that the problem of text-book correlations of algebra and trigonometry becomes of greater importance.

What contribution does Professor Brenke's book make towards the solution of the problem? A glance at the table of contents shows that the subject matter of plane trigonometry is inserted bodily between two chapters of algebra. The treatment of the trigonometry does not show traces of the existence of algebra to any greater extent than do ordinary treatments of the subject, neither do the chapters on algebra that follow show any influence of the trigonometry, except in those parts which are commonly studied after the trigonometry.

Clearly then, the process of correlating the two subjects has not yet gone very far. But, a first step has been taken; the two subjects have been brought within the same covers. In order to mix them more thoroughly, they will each have to be divided into smaller parts and brought into more frequent contact. If at the same time, we should consider how the work of the first year in college may be made to grow out of the high school work, we may come to a course better adapted to the needs of the students referred to above.

On the other hand, the book presents an interesting correlation of both the algebra and the trigonometry with the elements of analytic geometry and of the calculus. Introducing the graph of a linear function in Chapter IV, the author lets the graphical solution of simultaneous equations lead in Chapter V to an elementary discussion of the conic sections. In the first chapter on trigonometry the graphs of the functions are constructed, in contrast with most texts on trigonometry, in which they do not appear until near the end of the book. Although it is to be regretted that the graphical work is not developed further and not made use of more than for the solution of the equation

$$\sin 2\theta + \sin \theta + k = 0,$$

a method has at least been indicated. In Chapter XIII, Mac-laurin's series is established and applied at once to the computation of the trigonometric functions and of logarithms.

The treatment throughout the book is clear and to the point;

it speaks a language the student can easily understand. Numerous examples and problems are given with every topic. We welcome Chapter XIV on computation, approximations, differences, and interpolation. The requirements of practical computation are responsible for so many of the algebraic reductions with which our classes struggle, that a first hand experience with the problems of computation must help the students to see the reasons for some of the hardships inflicted upon them. The review in Chapters I and II of those parts of the high school work which are stumbling blocks for most freshmen and the discussion of division by 0 and of infinity will be appreciated by those who teach freshmen classes.

The definition of infinite series in § 196 confuses the notion of series with that of sequence. In § 246 we read "An arrangement of the numbers 1, 2, 3, ..., n is called an inversion," a statement which attributes to this word a meaning not usually given it.

Many of the subjects taken up require a more rigorous treatment than it would be wise to give them in an introductory text. In most such cases the finer points are not slurred over, but clearly indicated and then taken for granted, as, for instance, in discussing $\lim_{z \rightarrow \infty} (1 + 1/z)^z$, in deriving Maclaurin's series, etc.

The last sixty pages of the book contain the answers for the odd-numbered exercises, an index, and two appendices. The first of the appendices gives a list of formulas, definitions, and theorems. The second includes, besides the tables commonly found, also a conversion table for changing from sexagesimal to radian measure, and conversely; one for $\log_e x$, e^x , and e^{-x} from $x = 0$ to $x = 5$ at intervals of 0.05; and one for the squares and cubes, square and cube roots of numbers between 1 and 100. A handy cardboard protractor-ruler accompanies the book.

Among the misprints, we notice c in place of c^2 on page 23, 2d line from the top; and $\sqrt[4]{9.35}$ in place of $\sqrt[4]{9.35^3}$ on page 29, 10th line from the bottom.

ARNOLD DRESDEN.

College Algebra. By SCHUYLER C. DAVISSON, PH.D. New York, The Macmillan Co., 1910. Small 8vo. xvi + 243 pp. \$1.50 net.

IN this book no attempt at correlation is made; it is a pres-

entation of the topics usually taken up in the freshman class in algebra, different in many respects however from the usual text.

The preface states that "the book is not written for mathematicians, but for students desiring to know the elements of ordinary algebra." Can this be said to describe the state of mind of the majority of our college freshmen? Had we not better face our problem squarely and recognize the situation we have to deal with in our classes? As it appears to the reviewer, of those entering the freshman class in mathematics, not many have a desire to know the elements of ordinary algebra. At best, their mind is void of any desire with regard to algebra; if we can succeed in arousing their interest, we may be able to call it forth; but, to count on it at the beginning of the college course, seems too optimistic. It is for this reason that the book does not seem suitable for the ordinary freshman course, to which students are admitted who have had one and a half years of secondary algebra.

That the author seems to have had in mind students who have had more than one and a half years of preparatory work, or at least students who have to some extent found themselves with respect to algebra, would also appear from the following paragraph in the preface: "The text does not contain a very large number of problems, but it is believed to contain a sufficient number to illustrate the theory. The aim has been to give the student an idea that algebra consists of something more than problem solving." Assuming this to be the author's point of view, the present writer believes that we have here an excellent text before us.

Part I—entitled Numbers and written, as is Chapter XV on Limits, by Professor A. M. Kenyon—opens with a presentation of the system of natural numbers and the laws governing the operations upon them. In order to make the inverse operations of division, evolution, and subtraction possible within the domain, successive extensions of the number system are made, obtaining step by step the domains of positive rational numbers, positive real numbers, real numbers, and complex numbers. Few theorems are proved in this chapter. It seems that the presentation is somewhat difficult to follow for one who is making his first acquaintance with the subject. Greater unity might have been obtained if, the positive rational numbers having been introduced as pairs (a, b) of integers whose order relation is

defined by

$$(a, b) \leq (c, d) \text{ according as } ad \leq bc,$$

the negative rational numbers had been introduced as pairs (a, b) of integers whose order relation is given by

$$(a, b) \leq (c, d) \text{ according as } a + d \leq b + c.$$

The chapter would also have gained strength if the theorems concerning at least one of the domains considered had been stated and *proved* in terms of the original definition of the numbers of the domain as number pairs.

In § 40, the student is asked to show that the definition of addition for rational numbers is consistent with the one given for the natural numbers; similar exercises are set for the other operations and extensions. Would it not be well to emphasize more strongly the important point involved?

Part II — Algebra — opens with Chapter II on definitions, followed by Chapter III on fundamental laws and identities. The definition of an algebraic function in § 132 as “an algebraic combination of quantitative and operative symbols that has meaning” seems unsatisfactory. What is an “algebraic combination of symbols”? How can we tell whether such a combination has meaning, unless we know in what domain we are working? Would $2^2 - 3 + 5^3$ not be an algebraic function under this definition? Have we not here a confusion of the notions function and functional value? About variables and constants, § 133 says, among other things: “When several literal quantities are involved in a function, the first letters of the alphabet shall be considered as constants, while the last letters shall be considered as variables,” a statement which does not give any information whatever about the concept variable.

The remaining chapters, on factorization of rational integral functions; equations in one and more unknowns and determinants; indices, surds, and complex numbers; sequences; permutations and combinations; mathematical induction; undetermined coefficients and partial fractions; limits and infinite series; theory of equations; logarithms and variation, contain a very attractive treatment of these various topics, presenting several features with regard to subject matter as well as arrangement that are different from what is usual in a college algebra. Mention should be made of the following points:

The section on factorization of symmetric functions in Chapter IV; the determination of the integral roots of equations of higher degree than the second in Chapter V, rather than in the later chapter on the theory of equations; the treatment of determinants in the chapter on linear simultaneous equations and the application of determinants to the solution of systems of n linear homogeneous equations in n variables and of n linear non-homogeneous equations in n or in $n - 1$ variables.

The treatment of the theory of limits in Chapter XV should be noticed for its excellence; the substance of the ϵ, δ definitions of limits is given verbally, and illustrated graphically without using the algebraic statement of the definitions; the presentation of infinite sequences and series becomes clear as a result of this treatment of limits.

The exercises in verbally expressing the formulas used are a welcome aid in the efforts to combat the carelessness of expression to which our students are so largely accustomed. The method of graphing a function is explained in Chapter V, but is made very little use of in the later chapters. One misses the graphical illustrations particularly in the chapters on simultaneous equations and in the chapter on variation, while the graphical methods could have been used to greater advantage in the chapter on theory of equations and in the chapter on limits.

ARNOLD DRESDEN.

Spezielle algebraische und transzendente ebene Kurven, Theorie und Geschichte. Von GINO LORIA. Autorisierte nach dem italienischen Manuskript bearbeitete Ausgabe von F. SCHÜTTE. Zweite Auflage. Leipzig, B. G. Teubner. Erster Band: *Die algebraischen Kurven*, 488 pp. and 14 plates, 1910. Zweiter Band: *Die transzendenten und abgeleiteten Kurven*, 384 pp. and 6 plates, 1911.

THE value and importance of this work, which was first published in 1902, are partially indicated by this early appearance of a complete revision. About one hundred and twenty-five pages and three plates have been added. Since 1902 much has been done in the theory of special plane curves, and this work has, as far as possible, been incorporated in the new edition, which is now published in two volumes instead of one.

Many of the excellent features of the work were pointed out and discussed at some length in Professor E. B. Wilson's

review of the first edition.* As these have been retained unchanged in the revision, it is only necessary in the present notice to call attention to the principal additions to material and changes in arrangement.

The original edition contained extensive accounts of the curves of the third and fourth orders, but nothing on curves of the fifth order. Since then a good deal has been written on special curves of order five by Morley, Basset, Snyder, Field, and others, so that it is but natural to find in the revised work an account of these researches and of the theory and history of the general quintic. It may be noted that the theory of the general sextic is now included, in addition to the treatment of the special sextics given in the earlier edition.

In the review mentioned Professor Wilson discussed in detail what Loria called *panalgebraic curves*, and stated that for advanced students this section, which was then relegated to a note at the end of the volume, was of more interest and importance than all the rest of the book. The change of arrangement by which this discussion is brought to the dignified position of Chapter I seems to the reviewer most advantageous and important. In this position the subject is better situated to attract students, and before many years it may receive merited attention.

The other changes in the work are made principally for the purpose of bringing the older treatment up to date or to complete the history.

C. L. E. MOORE.

Encyklopädie der Elementar-Mathematik. Von HEINRICH WEBER und JOSEF WELLSTEIN. Erster Band: *Elementare Algebra und Analysis*. Bearbeitet von H. WEBER. Dritte Auflage. Leipzig, Teubner, 1909. xviii + 532 pp.

THE first edition of this book has been reviewed in this BULLETIN.† The third edition differs from the first primarily in the introduction of nearly 100 pages of new subject matter. These additions consist chiefly of historical notes, which are appended to a number of chapters, and an entirely new chapter at the end, which is devoted to the graphical representation of a function, differentiation, and integration.

The value of the book is much increased by the additions, which harmonize with the spirit of the earlier editions.

F. W. OWENS.

* BULLETIN, vol. 9 (1903), pp. 492-501.

† Vol. 10, pp. 200-204.

Die Determinanten. By EUGEN NETTO. Leipzig, Teubner, 1910. 8vo. vi + 129 pp. 3.60 marks.

THE enormous growth in recent years of both mathematics and engineering is nowhere shown better than in the appearance in both Europe and America of many small handbooks particularly designed to give in little space the main features of some definite field of these large subjects. This volume is of such a character. It undertakes to develop the elementary theory of determinants with some applications. The applications are not technical.

The first two chapters are devoted to the elementary properties and expansions. Then follows one on evaluation, and one on products. A chapter follows on arrays, which are here called matrices. Then comes a chapter on particular determinants, one on the solution of linear equations, and one on resultants. Following is a chapter on linear substitutions, which could have properly formed part of the chapter on matrices. In our opinion, it is a mistake to consider the array of symbols merely as an array. The composition of matrices, if not the mere ordering, makes the symbols stand for more than mere arrays. They are indeed n^2 -fold multiple quantities and should at least be treated as multiplexes if not as operators. From this point of view the linear substitution is merely one example of a matrix. A chapter is devoted to geometric applications. The book is closed by a very brief chapter on differentiation and one on functional determinants, which last however would have been labeled more correctly "The Jacobian." As a whole the treatment is clear, well illustrated, and all that could have been given in the space.

JAMES BYRNIE SHAW.

Die Theorie der Kräftepläne. By H. E. TIMERDING. Leipzig, Teubner, 1910. 8vo. vi + 99 pp. 3 marks.

THIS little volume is number 7 in Teubner's Mathematisch-physikalische Schriften. It undertakes to occupy a middle ground between the purely theoretical or geometrical side of graphical statics and the purely practical or engineering side. The object is to interest the engineer in a more general mathematical point of view and to show the mathematician that applications furnish very interesting developments of pure theory. Thus we find the null-system of Möbius used throughout to show the reciprocal relations of the force polygon and

the funicular polygon. Few detailed solutions are given, the emphasis being rather on the first principles and the essential similarity of the various methods in use. The space figure over the plane figures is utilized to assist the development.

Engineers who want detailed methods will find little here to interest them. Those who desire a knowledge of fundamentals which will enable them to devise their own methods will find this text very suggestive. Mathematicians will find here concrete examples of some geometrical transformations, but little that would serve to advance the theory of such transformations. Where the "finest flowers bloom along the way of application" the author scarcely shows.

JAMES BYRNIE SHAW.

Vorlesungen über technische Mechanik. Von AUGUST FÖPPL.
Band 6: *Die wichtigsten Lehren der höheren Dynamik.*
Leipzig, Teubner, 1910. xii + 490 pp.

VARIOUS volumes in various editions of Föppl's lectures on mechanics have been reviewed in this BULLETIN,* always with unstinted though measured praise. That the public shares our enthusiasm for the work is fully indicated by the rapidity with which new editions follow one another. The work as now constituted, in what may perhaps be believed its final form, has six volumes instead of the original four. The titles are Einführung in die Mechanik, Graphische Statik, Festigkeitslehre, Dynamik, Die wichtigsten Lehren der höheren Elastizitätstheorie, and Die wichtigsten Lehren der höheren Dynamik. It is this last which is now under review. But before we begin let us repeat from an earlier review the wish that our students of technology had the advantage of such a work as this in English. There are not nearly enough exercises and examples to suit our needs, but the text we believe is far superior to anything we have.

The first section of the work deals with relative motion. Now-a-days we hear a great deal about the principle of relativity, according to which all actual motions in our physical universe are representable by differential equations which are invariant under the transformations of the Lorentz group, that is, the orthogonal group which leaves the form $x^2 + y^2 + z^2 - ct^2$ invariant. This has served to emphasize the fact that the old-fashioned Newtonian mechanics has also its principle of rela-

* See vol. 9, p. 25, and vol. 13, p. 520.

tivity. Minkowski set the matter in a clear light in his lecture at Cologne in 1908;* recently Klein has been elaborating it;† both authors, however, are naturally more interested in the new relativity than in the old; both assume a considerable amount of mathematics unfamiliar to the technical student. Föppl begins his discussion of relative motion with a clear and searching treatment of the relativity of ordinary mechanics; these pages should be read by nearly every one. In the applications of relative motion to various problems the text follows Coriolis's theorem in much the usual manner.

The second part of the work deals with generalized coordinates and Lagrange's equations. There are numerous applications, chiefly to various sorts of pendulum motion and to the rolling wheel. The third part is on the top, just over an even hundred pages. This long development of a somewhat difficult subject is amply justified by the ever increasing practical use of the gyroscope. The Schlick apparatus for regulating gyrostatically the motion of ships receives especial mention; the Brennan and Scherl monorail cars are barely touched upon. The author very naturally gives some account of his own interesting experiments to obtain the angular velocity of the earth from observations on the motion of a "top." A short fourth section of the text, entitled various applications, is chiefly occupied with the theory of the governor with and without friction; a very short discussion of planetary motion is given. It will be noticed that Föppl sticks closely to the subjects which are of most interest and importance to the engineer.

The fifth and last section of the book is on hydrodynamics. Some of the subjects treated are: the irrotational flow of liquids in two dimensions with applications of the theory of functions of a complex variable, various kinds of water waves, the theory of vortices, the theory of the turbine, and the motion of viscous liquids with applications to the motion of a sphere and to frictions on journals. In view of the fast increasing interest in aeronautics some aerodynamics might advantageously have been added. Although the mathematics of this section and of this volume as a whole is somewhat more advanced than that of the first four volumes and might offer some difficulties to the general run of our undergraduate students of engineering even in our best schools, it is no more than should be required, and

* See the *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 18 (1909), p. 75.

† See *ibidem*, vol. 19 (1910), p. 281.

in some institutions is required, of students for the master's degree. In conclusion let us hope that Föppl's lectures will continue and grow in popularity not only in their home but abroad; their merits are of the highest; their use can but inure to the benefit of the user. Once more we wish we had this work or such a work in English.

E. B. WILSON.

CORRECTION.

ON pages 454, 455 of Professor Glenn's paper in the June BULLETIN, in formulas (6), (7), (8), for $-$ read $+$.

NOTES.

THE July number (volume 12, number 3) of the *Transactions of the American Mathematical Society* contains the following papers: "On properties of a domain for which any derived set is closed," by E. R. HEDRICK; "Important covariant curves and a complete system of invariants of a rational quartic curve," by J. E. ROWE; "An application of Moore's cross ratio group to the solution of the sextic equation," by A. B. COBLE; "On the use of the co-sets of a group," by G. A. MILLER; "The southerly deviation of falling bodies," by W. H. ROEVER; "An application of a (1-2) quaternary correspondence to the Kummer and Weddle surfaces," by VIRGIL SNYDER; "On semi-discriminants of ternary forms" by O. E. GLENN.

THE July number (volume 33, number 3) of the *American Journal of Mathematics* contains the following papers: "The rational plane quartic as derived from the norm-curve in four dimensions by projection and section," by J. R. CONNER; "On the analytical basis of non-euclidean geometry," by W. H. YOUNG; "Curves in non-metrical analysis situs with an application in the calculus of variations," by N. J. LENNES.

AT the meeting of the London mathematical society held on June 8 the following papers were read: By G. H. HARDY, "On the multiplication of Dirichlet's series"; by G. H. HARDY and J. E. LITTLEWOOD, "On the range of Borel's method for the summation of series"; by W. H. YOUNG, "On the con-

vergence of Fourier series and of the allied series"; by W. M. PAGE, "Two dimensional problems in electrostatics and hydrodynamics."

THE Verein zur Förderung des mathematischen und naturwissenschaftlichen Unterrichts will hold its twentieth general meeting at Münster, June 5 to 8, 1911, under the presidency of Professor A. THAER, of Hamburg.

THE fifty-first reunion of deutsche Philologen und Schulmänner will be held at Posen, October 3 to 6, 1911. Papers and addresses in the mathematical section will be in charge of Drs. SPIES and THIEME.

THE fifth international congress of mathematicians will be held at Cambridge, England; August 22 to 28, 1912. It will be organized by the Cambridge philosophical society and the London mathematical society.

THE second congress of Scandinavian mathematicians will be held at Copenhagen, August 28 to 31, 1911.

THE committee appointed by the St. Petersburg academy of sciences to arrange the manuscripts of Euler for the use of the Euler commission has now completed its investigation, and reports that valuable collections of letters and a number of unpublished notes and memoirs, together making 209 titles, have been found and forwarded to the editors in Zürich. The material fills seven large chests. The Euler letters and manuscripts in the possession of the Royal society have all been carefully copied, and the copies placed at the disposition of the commission. In order to make the publication as complete as possible, the chairman of the commission expresses the wish that all letters of Euler that may be in the possession of other societies or of individuals may be made accessible to the commission.

UNDER the general title of "Fortschritte der mathematischen Wissenschaften in Monographien," by O. BLUMENTHAL, the firm of B. G. Teubner, of Leipzig and Berlin announces three tracts that are to appear soon. The titles are "Grundlagen der Theorie der ganzen Funktionen unendlicher Ordnung," by O. BLUMENTHAL; "Untersuchungen über den sogenannten Picardschen Satz," by C. CARATHÉODORY; "Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen," by D. HILBERT.

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THE publishing house of Longmans, Green, and Company announces the early appearance of a volume of Monographs on topics of modern mathematics relevant to the elementary field, edited by J. W. A. YOUNG. Titles and authors are as follows: The foundations of geometry, by OSWALD VEBLEN; Modern pure geometry, by T. F. HOLGATE; Non-euclidean geometry, by F. S. WOODS; The fundamental propositions of algebra, by E. V. HUNTINGTON; The algebraic equation, by G. A. MILLER; The function concept and the fundamental notions of the calculus, by G. A. BLISS; The theory of numbers, by J. W. A. YOUNG; Constructions with ruler and compasses, regular polygons, by L. E. DICKSON; The history and transcendence of π , by D. E. SMITH.

THE mathematical society and the society of applied physics of the University of Göttingen have given 100,000 marks toward founding an institute of mathematics at the University. Two manufacturing firms have given 50,000 marks for the same purpose.

ON October 27, 1910, the mathematical collection of the library of the University of Toulouse was destroyed by fire. In so far as books and manuscripts are concerned that can be replaced, the work of restoring the library has already begun. However, as the loss is much greater than can be covered by its available income, the University makes the general appeal to other universities, learned societies, libraries, publishers, and authors to send to it such duplicate copies as can be spared. In consideration of the valuable services to science rendered by the *Annales de la Faculté des Sciences de Toulouse* it is hoped that the appeal may meet with a generous response.

The following courses are announced for the academic year 1911-1912: CLARK UNIVERSITY.—By Professor W. E. STORY: Analytic geometry of higher plane curves, higher surfaces, and twisted curves, three hours; Hyperspace and non-euclidean geometry, two hours; History of mathematics, two hours, first half-year; Algebraic invariants, two hours, second half-year; Seminary.—By Professor H. TABER: Differential equations and calculus of variations, five hours; Transformation groups, two hours, first half-year; Picard-Vessiot theory of linear differential equations, two hours, second half-year; Seminary.—By Professor A. G. WEBSTER: The partial dif-

ferential equations of mathematical physics.—By Mr. J. DE PEROTT: Theory of numbers, two hours, first half-year; Algebraic substitutions, two hours, second half-year.—By Dr. F. B. WILLIAMS: Conic sections and quadric surfaces, two hours.

UNIVERSITY OF PENNSYLVANIA.—By Professor E. S. CRAWLEY: Solid analytic geometry, two hours; Higher plane curves, three hours.—By Professor G. E. FISHER: Differential equations, two hours; Theory of functions, three hours.—By Professor I. J. SCHWATT: Theory of functions of a real variable, three hours; Advanced calculus, three hours.—By Professor G. H. HALLETT: Introduction to modern higher algebra, three hours, first half-year; Galois theory of equations, three hours, second half-year; Finite groups, three hours.—By Professor F. H. SAFFORD: Mathematical theory of precision of measurements, three hours, first half-year; Curvilinear coordinates, three hours, second half-year.—By Professor O. E. GLENN: Theory of invariants, three hours.—By Dr. A. M. HILTEBEITEL: Theoretical mechanics, three hours.—By Dr. H. H. MITCHELL, Theory of linear groups, three hours.

PROFESSOR E. PICARD, of the University of Paris, has been elected a foreign member of Belgian academy of sciences.

PROFESSOR F. D'ARCAIS, of the University of Padua, has been elected a member of the royal institute of Venice.

DR. L. TONELLI has been appointed docent in the calculus at the University of Bologna.

DR. W. BLASCHKE, of the University of Bonn, has been appointed docent in mathematics at the University of Greifswald.

DR. F. BERNSTEIN, of the University of Göttingen, has been promoted to an associate professorship of mathematics.

DR. G. DU PASQUIER has been appointed professor of mathematics at the University of Neuchâtel.

AT Cornell University, Dr. D. C. GILLESPIE has been promoted to assistant professorship of mathematics. Mr. F. M. MORGAN has been appointed assistant in mathematics.

Dr. L. I. NEIKIRK, of the University of Illinois, has been appointed instructor in mathematics in the University of Washington.

MR. F. W. BEAL, of Brown University, has been appointed instructor in mathematics in the University of Pennsylvania.

MR. R. D. FULLERTON has been appointed lecturer in mathematics at McGill University.

PROFESSOR E. V. HUNTINGTON, of Harvard University, has been granted a year's leave of absence; he will spend the first half-year abroad.

AT the University of California, Professor C. A. NOBLE has been granted leave of absence for the first term of the academic year 1911-1912; Professor D. N. LEHMER has been granted leave of absence for the second term.

PROFESSOR T. W. D. WORTHEN, of Dartmouth College, has resigned to enter the government service.

CATALOGUES of second-hand books: Gustav Fock, Leipzig, Schlossgasse 7-9, Antiquariats Katalog 391, about 4000 titles in pure and applied mathematics.—A. Hermann, 6 rue de la Sorbonne, Paris V, catalogue 107, about 1600 titles in pure and applied mathematics.—W. Heffer and Sons, Cambridge, England, catalogue 75, about 900 titles in pure and applied mathematics.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

- AUERBACH (F.) und ROTHE (R.). Taschenbuch für Mathematiker und Physiker. Leipzig, Teubner, 1911. 8vo. 9+567 pp. Cloth. M. 7.00
- BEUTEL (E.). Algebraische Kurven. 2ter Teil: Theorie der Kurven 3ter und 4ter Ordnung. (Sammlung Göschen. 436.) Leipzig, Göschen, 1911. 8vo. 136 pp. Cloth. M. 0.80
- CHATELET (A.). Sur certains ensembles de tableaux et leur application à la théorie des nombres. Paris, Gauthier-Villars, 1911. 4to. 104 pp.
- DUMONT (E.). Grandeurs et nombres. Arithmétique générale. Définitions et propriétés fondamentales des grandeurs géométriques et de leurs mesures: nombres, naturels, qualifiés, complexes, ternions, et quaternions. Paris, Hermann, 1911. Fr. 16.00
- ENCYCLOPÉDIE des sciences mathématiques pures et appliquées. Edition française. Tome I: Algèbre. Fascicule 3. Leipzig, Teubner, 1911. 8vo. pp. 329-424. M. 3.00
- ENRIQUES (F.). I numeri e l'infinito. (Scientia.) Bologna, Zanichelli, 1911. 8vo. 26 pp.
- HARTWIEG (P.). Konstruktion der Hauptachsen des Ellipsoides aus drei konjugierten Durchmessern. (Diss.) Strassburg, 1910.
- HEUSCHMANN (J. C.). Aus Laienmund ein Wort zum sogenannten grossen Fermatschen Satz. Erlangen, Junge, 1911. M. 2.00
- KARNASCH. Beweis II für den Fermatschen Satz, dass die Gleichung $x^n + y^n = z^n$ in ganzen Zahlen nicht lösbar ist, wenn $n > 2$. Zweite ergänzte Ausgabe. Berlin, 1911. M. 1.20
- KILL (P.). Beiträge zum Fundamentalproblem der Flächentheorie. (Diss.) Strassburg, 1910.
- KLEIN (F.). Elementarmathematik vom höheren Standpunkte aus. Iter Teil: Arithmetik, Algebra, Analysis. Vorlesung, gehalten im Wintersemester 1907-08. Ausgearbeitet von E. Hellinger. 2te Auflage Leipzig, Teubner, 1911. 8vo. 8+614 pp. M. 7.50
- KOENIGS (G.). Mémoire sur les courbes conjuguées dans le mouvement relatif le plus général de deux corps solides. (Mémoires présentés par divers savants à l'Académie des sciences de l'Institut de France. Extrait du tome 35, No. 1.) Paris, 1910. 4to. 219 pp.
- MACMAHON (P. A.). Memoir on the theory of the partition of numbers. Part 5: Partitions in two-dimensional space. London, Dulau, 1911. 4to. Sewed. 2 s.
- MIKIEWICZ (B.). Die Durchführbarkeit der Gleichung $a^n + b^n = c^n$ in ganzen Zahlen oder das sogenannte Fermat'sche Problem. Krakau, Friedlein, 1910. 8vo. 28 pp. M. 2.00
- MINKOWSKI (H.). Gesammelte Abhandlungen. Unter Mitwirkung von A. Speiser und H. Weyl herausgegeben von D. Hilbert. Zweiter Band. Leipzig, Teubner, 1911. M. 16.00

- PELLIZZARI (N.). Trasformazioni delle superfici applicabili sul caten-
oide ordinario allungato ed accordato. Pisa, Nistri, 1910. 8vo. 32 pp.
- PUCCHINI (A.). Le configurazioni regolari n_4 di punti e piani contetraedri.
Padova, Società cooperativa tipografica, 1910. 8vo. 16 pp.
- ROTHER (R.). See AUERBACH (F.).
- UYEN (M. J.). Algebraische Strahlencongruenzen und verwandte com-
plexe Ebenen als Schnitte derselben. Amsterdam, Muller, 1911. 8vo. 527 pp. M. 15.00

II. ELEMENTARY MATHEMATICS.

- BAILLIETT (S. J. D.). Nature's symphony; or lessons in number vibration.
Atlantic City, 1911. 8vo. 10+127 pp. \$1.25
- BARONI (E.). Trigonometria piana e sferica, ad uso delle scuole second-
arie superiori. (Biblioteca scolastica.) Firenze, Bemporad, 1910. 16mo. 4+304 pp. L. 2.75
- BAUER (W.) and HANXLEDER (E.). Lehrbuch der Mathematik zum
Gebrauche an Lyzeen. Drei Bände. Braunschweig, Vieweg, 1911. 8vo. Cloth. M. 11.00
- BISSON-MINIO (E.). Errori ed inesattezze piu comuni in aritmetica e
geometria, elenco di forme errate, con correzione e discussione, ad
uso degli allievi maestri. 2a edizione. Belluno, Cavessago, 1910. 16mo. 19 pp. L. 0.30
- BLODGETT (M. A.). New exercises and problems in elementary algebra,
for use as supplementary material in addition to regular text-book
work. Minneapolis, Northwestern School Supply Co., 1911. 12mo. 3+89 pp. \$0.25
- BRENKE (W. C.). A text-book on advanced algebra and trigonometry.
With tables. New York, Van Nostrand, 1910. 352 pp. Cloth. \$2.00
- COMBEROUSSE (C.). Cours de mathématiques à l'usage des candidats
à l'Ecole polytechnique. 4e édition, refondue et augmentée. Tome 3:
algèbre supérieure. 1re partie. Paris, Gauthier-Villars, 1911. 8vo. 21+768 pp. Fr. 15.00
- DAVISON. Algebra for secondary schools. With answers. London,
Cambridge University Press, 1911. 8vo. 324 pp. 3s.
- HALL (H. S.). A school algebra. Parts 1 and 2, in one volume. London,
Macmillan, 1911. 8vo. 3s. 6d.
- HAMMER (E.). Lehrbuch der elementaren praktischen Geometrie. Band
1: Feldmessen und Nivelieren. Leipzig, Teubner, 1911. 8vo. 20+766 pp. Cloth. M. 24.00
- HANXLEDER (E.). See BAUER (W.).
- HAWKES (H. E.), LUBY (W. A.) and TOUTON (F. C.). Second course
in algebra. Boston, Ginn, 1911. 12mo. 7+264 pp. \$0.75
- JAVELOT (R.). Des procédés pour résoudre les problèmes de géométrie
élémentaire, suivis des premiers éléments de géométrie analytique
appliqués à ces problèmes. Rennes, Oberthur, 1911. 8vo. 136 pp.

- KNOBLOCH (W.) und KÜHL (H.). Sammlung geometrischer Konstruktionen zum Gebrauche in Gewerbeschulen, Präparanden-Anstalten, Volks- und Mittelschulen. Altona-Ottensen, St. Carstens, 1911. 8vo. 87 pp. M. 1.20
- KÜHL (H.). See KNOBLOCH (W.).
- LAAGER (F.). Vereinfachter Lehrgang der Elemente der Trigonometrie. Zurich, Minerva, 1911.
- LIETZMANN (W.). Berichte und Mitteilungen, veranlasst durch die Internationale Mathematische Unterrichtskommission. V. Leipzig, 1911. M. 0.60
- LUBY (A. W.). See HAWKES (H. E.).
- MAHLERT (A.). See MÜLLER (H.).
- MILNE (W. J.). First year algebra. New York, American Book Co., 1911. 12mo. 320 pp. \$0.85
- MÜLLER (H.) und MAHLERT (A.). Mathematisches Lehr- und Übungsbuch für höhere Mädchenschulen. 2ter Teil: Planimetrie und Körperberechnungen. 4te Auflage. Leipzig, Teubner, 1911. 8vo. 7+122 pp. M. 1.80
- NOZIONI di aritmetica e geometria, con una raccolta di 1940 esercizi e problemi, per le classi V e VI elementari. 2a edizione. Roma, Romana, 1911. 16mo. 8+115 pp.
- PERROTTI (F.). Algebra, nozioni preliminari, ad uso del terzo corso della scuola tecnica di Sulmona. Sulmona, Angeletti, 1911. 8vo. 18 pp.
- SLAUGHT (H. E.) and LENNES (N. J.). Solid geometry; with problems and applications. Boston, Allyn & Bacon, 1911. 12mo. 6+190 pp. Cloth. \$0.75
- SMITH (E. R.). Plane geometry developed by the syllabus method. New York, American Book Co., 1911. 12mo. 192 pp. Cloth. \$0.75
- SUPPANTSCHITSCH (R.). Leitfaden der darstellenden Geometrie für die V bis VII Klasse der Realschulen. Wien, Tempsky, 1911. M. 4.00
- TESTI (G. M.). Corso di matematiche, ad uso delle scuole secondarie superiori e più specialmente degli istituti tecnici. Volume IV. Complementi di geometria. 3a edizione, riveduta e corretta. Livorno, Giusti, 1911. 8vo. 13+236 pp. L. 3.00
- TOUTON (F. C.). See HAWKES (H. E.).
- WANACH (B.). Tafel der Werte $\frac{ab}{a+b}$ für alle zweistelligen Werte von a und b zur Berechnung der Gewichte von Summen, Differenzen, Mittelwerten, und so weiter. (Veröffentlichung des Königlichen Preussischen Geodatischen Institutes. Neue Folge. 46.) Leipzig, Teubner, 1910. M. 1.20
- WEISBACH (J.). Tafel der vielfachen sinus und cosinus, sowie der vielfachen sinus versus von kleinen Winkeln, nebst Tafel der einfachen Tangenten, zum Gebrauche für Praktiker, Geometer und Mechaniker überhaupt und für Markscheider besonders. Berlin, Weidmann, 1911. 8vo. 28 pp. M 1 00

III. APPLIED MATHEMATICS.

- ANSCHUTZ. The history, description, theory, and practice of the gyro-compass. London, Rees, 1911. 8vo. 3s. 6d.
- BELL (C. E.). Applications of rates. Part 1. Chicago, La Salle Extension University, 1911. 4to. \$0.25
- BOTTONE (S. R.). Wireless telegraphy and hertzian waves. (Whittaker's library of arts, sciences and industries.) 4th edition. Revised. New York, Whittaker, 1910. 12mo. 12+136 pp. \$1.00
- BREU (W.). Three-phase transmission. A practical treatise on the economic conditions governing the transmission of electric energy by underground and overhead conductors. New York, Van Nostrand, 1911. 186 pp. Cloth. \$2.00
- CHASE (W. A.). Higher accountancy, principles and practice. 2 volumes. Chicago, La Salle Extension University, 1911. 8vo. \$6.00
- CLAYTON (W. B.) and CRAIG (J. W.). General treatise on electricity and electrical apparatus. New York, Y. M. C. A., 1910. 16mo. 357 pp. Flexible leather. \$3.00
- CRAIG (J. W.). See CLAYTON (W. B.).
- DRUDE (P.). Précis d'optique publié d'après l'ouvrage de P. Drude, refondu et complété par M. Boll. Tome 1er: Optique géométrique. Optique ondulatoire. Paris, Gauthier-Villars, 1911. 8vo. 10+376 pp. Fr. 12.00
- DUHEM (P.). Traité d'énergétique ou de thermo-dynamique générale. Tome 1er: Conservation de l'énergie. Mécanique rationnelle. Statique générale. Déplacement de l'équilibre. Paris, Gauthier-Villars, 1911. 8vo. 532 pp. Fr. 18.00
- FLEMING (J. A.). The principles of electric wave telegraphy and telephony. Second edition, revised and extended. New York, Van Nostrand, 1911. 924 pp. Cloth. \$7.50
- IGNATOWSKY (W. v.). Die Vectoranalysis und ihre Anwendung in der theoretischen Physik. 2ter Teil. Leipzig, Teubner, 1910. 8vo. 4+123 pp. Cloth. M. 3.00
- INCHLEY (W.). See MORLEY (A.).
- KARAPETOFF (V.). The electric circuit. New York, McGraw-Hill, 1911. 8vo. 85 pp. \$1.50,
- KÖRBER's Strahlendiagramm zur vereinfachten Herstellung Perspektivischer Zeichnungen. Zum Gebrauch für Architekten, Ingenieure Kunstgewerbetreibende und Landschaftsgärtner. 3te Auflage. Berlin, Ernst, 1911. 5+104 pp. M. 2.00
- LORENZ (H.). Die Theorie in der Technik. Rede zur Feier des Geburtstages Seiner Majestät des Kaisers und Königs gehalten am 27 Januar 1911 in der Aula der Königlichen Technischen Hochschule zu Danzig. Danzig, Schwital and Rohrbeck, 1911.
- MARQUARDT. Beiträge zur mathematischen Geographie. Eilenburg, Offenbauer, 1911. 8vo. 20 pp. M. 0.20
- MEAD (D. W.). Water power engineering. Corrected edition. New York, McGraw, 1911. 8vo. 803 pp. \$6.00
- MEMME (F. DE). Sulla struttura elicotetraedrica dei cristalli romboedrici. Genova, Pellas, 1911. 8vo. 8 pp.

- MORLEY (A.) and INCHELEY (W.). Elementary applied mechanics. London, Longmans, 1911. 8vo. 390 pp. 3s.
- PAILLOT (R.). Cours d'électricité théorique. Courants alternatifs. (Institut industriel des Nord. Génie civil, 2e année.) Lille, Janny, 1911. 4to. 138 pp.
- RICHARD (P. J.). Étude sur l'assurance complémentaire de l'assurance sur la vie avec de nombreux développements sur les assurances contre la maladie et l'invalidité. Paris, Hermann, 1911. Fr. 3.50
- ROUMAJON (J.). Mécanique. (Bibliothèque des écoles pratiques de commerce et d'industrie.) Paris, Delagrave, 1911. 18mo. 7+396 pp. Fr. 3.75
- ROU (J. W.). Steam turbines. New York, McGraw-Hill, 1911. 8vo. 143 pp. \$2.00
- THOMSON (W.). Mathematical and physical papers. Volume 5. London, Cambridge University Press, 1911. 8vo. 618 pp. 18 s.
- WALKER (G. T.). Outlines of the theory of electromagnetism. London, Cambridge University Press, 1911. 8vo. 60 pp. 3s.
- YORIMOTO-TASHI. L'énergie en douze leçons. Traduit du japonais, commenté par B. Dangennes. Tours, Arrault, 1911. 8vo. 128 pp.

TWENTIETH ANNUAL LIST OF PAPER

READ BEFORE THE AMERICAN MATHEMATICAL SOCIETY
SUBSEQUENTLY PUBLISHED, INCLUDING REFERENCES
TO THE PLACES OF THEIR PUBLICATION.

- BATES, W. H. An Application of Symbolic Methods to the Mean Curvatures in Hyperspace. Read (Chicago) Jan. Dec. 31, 1909; and Dec. 29, 1910. *Transactions of the Mathematical Society*, vol. 12, No. 1, pp. 19-38; Jan., 1911.
- BROCK, H. Ein Seitenstück zur Moebius'schen Geometrie der Wandschaften. Read Sept. 7, 1910. *Transactions of the Mathematical Society*, vol. 11, No. 4, pp. 414-448; Oct., 1911.
- BIRKHOFF, G. D. General Theory of Linear Difference Equations. Read Sept. 6, 1910. *Transactions of the American Mathematical Society*, vol. 12, No. 2, pp. 243-284; April, 1911.
- BLOCHFELD, H. F. On the Order of Linear Homogeneous Groups (Paper). Read (San Francisco) Sept. 24, 1910. *Transactions of the American Mathematical Society*, vol. 12, No. 1, pp. 39-42.
- BLISS, G. A., and MASON, MAX. Fields of Extremals in Space. Read Sept. 10, 1908. *Transactions of the American Mathematical Society*, vol. 11, No. 3, pp. 325-340; July, 1910.
- BÖRGER, R. L. On the Determination of the Ternary Modular Functions. Read (Southwestern Section) Dec. 1, 1906. *American Mathematical Monthly*, vol. 32, No. 3, pp. 289-298; July, 1910.
- CAJORI, F. Fourier's Improvement of the Newton-Raphson Approximation Anticipated by Murraille. Read Sept. 1910. *Bibliotheca Mathematica*, ser. 3, vol. 11, No. 2, pp. 132-137;
- Horner's Method of Approximation Anticipated by Ruffini. Read (Southwestern Section) Nov. 26, 1910. *Bulletin of the Mathematical Society*, vol. 17, No. 8, pp. 409-414; May, 1911.
- Historical Note on the Newton-Raphson Method of Approximation. Read (Chicago) Dec. 29, 1910. *American Mathematical Monthly*, vol. 18, No. 2, pp. 29-32; Feb., 1911.
- CARMICHAEL, R. D. Linear Difference Equations and their Solutions. Read Sept. 6, 1910. *Transactions of the American Mathematical Society*, vol. 12, No. 1, pp. 99-134; Jan., 1911.
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